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On a Stable Method for Option Pricing: Discontinuous Petrov-Galerkin Method for Option Pricing and Sensitivity Analysis

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Discontinuous Petrov–Galerkin (DPG) methodology, introduced by Demkowicz and Gopalakrishnan in their seminal work Demkowicz and Gopalakrishnan (2010), has become an integral tool in the computational mechanics field. In this paper, we extend the application of the DPG method to address a new domain, namely, the pricing of financial options and their sensitivity analysis within the framework of the Black-Scholes model. We present an innovative adaptation of the DPG method, incorporating both its primal and ultraweak formulations, to analyze an array of option types: Vanilla, American, Asian, and Barrier options. This robust mathematical framework provides a comprehensive toolkit for understanding and dissecting these complex financial instruments. A set of numerical experiments is carried out to assess the method's convergence, stability, and efficiency when applied to each option type independently. These rigorous tests serve to validate the effectiveness of our approach and attest to its suitability for tackling various option pricing challenges.

Keywords: Discontinuous Petrove-Galekin; Option Pricing; Variational Inequlity; Non–smooth Newton's Method; Sensitivity Anlaysis;

JEL Classification: Please provide at least one JEL Classification code

1. Introduction

Since their discovery in 1970, the option pricing formula developed by Black and Scholes Black and Scholes (1973) and Merton Merton (1973), known as the classical Black-Scholes (or BS-Merton) model, has garnered significant attention from academia and practitioners. This formula, which earned Black and Scholes the Nobel Prize in Economics Ferreyra (1998), serves as a fundamental tool for pricing options and has become widely utilized by investors as a means to devise risk-protected strategies against fluctuations in the price of underlying assets. Moreover, market makers rely on option price sensitivities, commonly known as Greeks, to design optimal hedges for their positions.

However, obtaining analytical solutions for pricing financial instruments, with a few exceptions, remains elusive. Among these instruments, exotic options, which are path-dependent derivatives, present particularly challenging valuation problems. American-style options, Asian options, and Barrier options exemplify the complexity associated with pricing these types of hedging devices, making analytical pricing infeasible.

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Consequently, researchers have endeavored to develop efficient numerical methods for option pricing as a natural remedy since the inception of the Black-Scholes model in 1970. Noteworthy among these methods are analytical approximations Barone-Adesi and Whaley (1987), Geske and Johnson (1984), stochastic mesh methods Broadie *et al.* (2004), Monte Carlo methods Boyle *et al.* (1997), Boyle (1977), Acworth *et al.* (1998), lattice-based methods employing finite element and finite difference techniques for solving corresponding partial differential equations Achdou and Pironneau (2005), Chiarella *et al.* (2014), Duffy (2013), Seydel and Seydel (2006), Tavella and Randall (2000), as well as mesh-free methods Kim *et al.* (2014), Fasshauer *et al.* (2004), Bastani *et al.* (2013).

Among these numerical methods, the weighted residual methods, specifically Galerkin methods, have consistently captivated the interest of the scientific community Seydel and Seydel (2006), Achdou and Pironneau (2005) due to their undeniable merits in solving differential equations. Galerkin methods possess numerous advantages, including an elaborate and comprehensible theory for prior and posterior error estimation. Consequently, these methods find particular relevance in quantitative finance, where path-dependent options can benefit from well–understood error estimates to adaptively refine the mesh in their respective domains. Examples where the strengths of variational methods can be readily exploited include American options near the optimal exercise boundary or multi-factor options with intricate domains Seydel and Seydel (2006), Achdou and Pironneau (2005).

The Black-Scholes partial differential equation (PDE), representing a time-dependent parabolic PDE of the convection-diffusion type, stands as the state-of-the-art model in option pricing. However, it is well–documented in the literature (Ern and Guermond (2004), Strang *et al.* (1974), and Douglas and Russell (1982), among others) that this class of problems can exhibit numerical instability when the coercivity condition is violated, often due to the small coefficients associated with the second-order differential operator. This instability can manifest as a loss of accuracy or oscillatory behavior in the computed solution.

In response to these challenges, the discontinuous Petrov-Galerkin (DPG) method with optimal test space was developed by Demkowicz and Golapalakrishnan Demkowicz and Gopalakrishnan (2010). Since its introduction, the DPG method has found extensive use in the numerical solution of differential equations, particularly convection-dominated diffusion problems Demkowicz and Heuer (2013), Ellis *et al.* (2016), Chan *et al.* (2014), Chan (2013), as well as PDE-constraint optimization problems Bui-Thanh and Ghattas (2014), Causin and Sacco (2005) encountered in computational mechanics.

The DPG method is designed with an optimal test space, which is distinct from the trial space, through a projection onto a continuous space. This design ensures the continuity and coercivity of the discrete scheme, provided that the test and trial spaces meet certain regularity conditions across any mesh. Additionally, the method incorporates a built-in error indicator that supports automatic adaptivity.

Viewed as a minimum residual method, the DPG method consistently yields a symmetric (Hermitian) positive definite stiffness matrix. This feature is particularly beneficial for developing iterative solution algorithms, such as those used for solving variational inequalities in pricing American-type options.

In this paper, motivated by the unconditional stability and solid mathematical theory of the DPG method, we propose its application to the problem of option pricing and Greeks estimation within the Black-Scholes model. Specifically, we present both ultraweak and primal formulations of the DPG method for pricing Vanilla options, American options, Asian options, and double Barrier options, as well as their sensitivity analysis. We focus on the time-independent DPG method and employ a time-stepping strategy to solve the problem over time. Additionally, we introduce a graph norm for each problem, in which the optimal test space is established, and assess the efficiency of the proposed methods through various numerical examples. In the case of American option pricing, we extend the DPG method to tackle both associated the free boundary value problem and the linear complementarity problem, enabling us to obtain the early exercise boundary.

To mitigate the computational expense associated with evaluating the optimal test space using the test-to-trial operator introduced in the original mathematical theory of the method Chan (2013), Roberts (2013), we leverage a broken test space that localizes the evaluation of the optimal test space on each element, thereby ensuring element-wise conformity. Employing the method with a discontinuous optimal test space allows for parallelization of the computation assembly and local computation of the test space, rendering the method reliable and viable. This feature of the DPG method facilitates the development of high-performance implementations and harnesses the capabilities of highly parallel computers.

It is important to note that our intention is not to compete with previous numerical schemes employed in the literature, despite the many desirable aspects of the DPG method. Instead, by encouraging a broader adoption of this method among researchers, we believe its unique features will prove invaluable in tackling more complex and challenging problems in quantitative finance, including option pricing in higher dimensions beyond one-dimensional scenarios, and valuing securites in more complex models.

The remainder of this paper is organized as follows. Section 2 provides a concise introduction to the Discontinuous Petrov-Galerkin method with optimal test space, while Section 3 establishes the notation and elementary tools from functional analysis. In Section 4, we present the DPG method for Vanilla options and introduce the graph norm for both primal and ultraweak formulations. Convergence analysis for European option pricing is performed in this section. Section 5 focuses on the pricing of exotic options, including American options, Asian options, and double Barrier options, by introducing the corresponding graph norms. We provide numerical solutions for standard examples found in the literature. Finally, in Section 6, we present the DPG method for sensitivity analysis of option pricing problems and evaluate the performance of the DPG method in computing Greeks for both exotic and vanilla options.

2. The DPG Method

In this section, we provide a high-level introduction to the Discontinuous Petrov-Galerkin (DPG) method with Optimal Test Functions. We begin with a brief review of the method for the steady-state problem, and in Section 4, we present a more concrete definition of the spaces to address the option pricing problem.

We start with the standard well–posed abstract variational formulation, which may not necessarily have a symmetric functional setting. The goal is to find $u \in U$ such that

$$b(u, v) = l(v), \quad v \in V, \tag{1}$$

where the trial space U and test space V are appropriate Hilbert spaces. The functional $l(\cdot)$ is a continuous linear functional, and $b(\cdot, \cdot)$ is a bilinear (sesquilinear) form that satisfies the inf-sup condition:

$$\sup_{v \in V} \frac{|b(u, v)|}{|v|_V} \geq \gamma |u|_U, \quad \forall u \in U, \tag{2}$$

ensuring the well–posedness of the variational form (1). The discretized version of this variational form with the Petrov-Galerkin method seeks to find $u_h \in U_h \subset U$ such that

$$b(u_h, v_h) = l(v_h), \quad v_h \in V_h. \tag{3}$$

According to Babuška's theorem Babuška (1971), for a discretized system (3) where $\dim(U_h) = \dim(V_h)$, stability, or in other words well–posedness, is achieved if the discrete inf-sup condition is satisfied:

$$\sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{|v_h|_V} \geq \gamma_h |u_h|_U, \quad \forall u_h \in U, \tag{4}$$

where the inf-sup constant γ_h must be bounded away from zero, i.e., $\gamma_h \geq \gamma > 0$. Selecting the discrete spaces for the trial and test spaces is crucial. The trial space U_h is usually chosen based on approximability, while the test space V_h can be designed to possess specific properties of the numerical algorithm, such as ensuring well-posedness.

The DPG method with Optimal Test Functions is designed to find, for each discrete function u_h from the trial space U_h, a corresponding optimal test function $v_h \in V$ that acts as a supremizer of the inf-sup condition. In other words, the optimal test function $v_h \in V$ satisfies

$$\sup_{v \in V} \frac{|b(u, v)|}{|v|_V} = \frac{|b(u, v_h)|}{|v_h|_V}. \tag{5}$$

To achieve this, we introduce a trial-to-test operator $T : U \longrightarrow V$. The optimal test space is defined as the image of the trial space under this operator, i.e., $V_h^{\text{opt}} := T(U_h)$. The function $v^{\text{opt}} \in V_h^{\text{opt}}$ satisfies

$$(v^{\text{opt}}, v)_V = (Tu_i, v)_V = b(u_i, v), \quad \forall v \in V, \tag{6}$$

where $(\cdot, \cdot)_V$ is the inner product on the test space. The equation (6) uniquely determines the optimal test space using the Riesz representation theorem, and the discrete stability of the form (3) is automatically guaranteed. The test function defined in (6) is designed such that satisfying the supremizer of the continuous inf-sup condition implies satisfaction of the discrete inf-sup condition, ensuring discrete stability. Moreover, we have

$$\sup_{v_h \in V_h^{\text{opt}}} \frac{|b(u_h, v_h)|}{|v_h|_V} \geq \frac{|b(u_h, Tu_h)|}{|Tu_h|_V} = \sup_{v \in V} \frac{|b(u_h, v)|}{|v|_V} \geq \gamma |u_h|_U, \tag{7}$$

which implies that the inf-sup constant $\gamma_h \geq \gamma$.

THEOREM 2.1 *The trial-to-test operator $T : U \longrightarrow V$ is defined as follows:*

$$Tu = R_V^{-1} Bu, \quad u \in U, \tag{8}$$

where $R_V : V \longrightarrow V'$ is the Riesz operator corresponding to the test inner product. The linearity of T can be easily shown.

Proof. see Demkowicz (2020). □

It has been shown in Demkowicz (2020) that the Ideal Petrov-Galerkin method, which was introduced above, is equivalent to a mixed method and a minimum residual method, where the residual is defined in a dual norm. The Ideal PG method provides a built-in error indicator for mesh adaptivity through the corresponding mixed method, where Riesz's representation of the residual in the dual test norm is exploited. Consider ϵ as the solution of the following variational form for a given $u_h \in U_h$:

$$(\epsilon, v)_V = l(v) - b(u_h, v), \quad \forall v \in V. \tag{9}$$

The Riesz representation of the residual ϵ is uniquely determined by (9). The following mixed problem can be defined:

$$\begin{cases} u_h \in U_h, \quad \epsilon \in V, \\ (\epsilon, v)_V + b(u_h, v) = l(v), \quad v \in V, \\ b(\delta u_h, \epsilon) = 0, \quad \delta u_h \in U_h, \end{cases} \tag{10}$$

where the solution of the Ideal Petrov-Galerkin problem with optimal test space can be obtained by solving the mixed Galerkin problem (10). Thus, the method inherently provides a built-in a posteriori error indicator ϵ, measured in the test norm.

However, determining the optimal test functions analytically, except for some simple model problems, is impossible. Therefore, approximating the optimal test space to satisfy the discrete inf-sup condition is necessary. An enriched test subspace $V_h \subset V$ is employed to address this approximation. Thus, the Practical Petrov-Galerkin method with approximated optimal test space can be obtained as follows:

$$\begin{cases} u_h^r \in U_h, \\ \\ b(u_h^r, T^r \delta u_h) = l(T^r \delta u_h), \quad \delta u_h \in U_h, \end{cases} \tag{11}$$

where the approximated optimal test space is computed with components that satisfy the standard discretization

$$\begin{cases} T^r u \in V^r, \\ \\ (T^r u, \delta u_h)_V = b(u, \delta v), \quad \delta v \in V^r. \end{cases} \tag{12}$$

By increasing the dimension of the discrete enriched test space, we ensure the satisfaction of the discrete inf-sup condition for the system (3). This strategy is valid based on Brezzi's theory Demkowicz (2020), which allows the dimension of the discrete test space V^r to exceed the dimension of the trial space, contrary to Babuška's theory, which requires overlapping dimensions for the trial and test spaces. The stability reduction in the practical Petrov-Galerkin method can be analyzed using Fortin operators Gopalakrishnan and Qiu (2014), Nagaraj *et al.* (2017).

Despite the numerous advantages of the practical Petrov-Galerkin method, the computational cost of determining the optimal test space globally through the operator T is high. By employing a broken test space, this issue can be addressed by localizing the evaluation of the optimal test space on an element-wise basis. This approach justifies the name "Discontinuous" Petrov-Galerkin method (DPG) with optimal test functions. However, the introduction of a broken test space requires the incorporation of additional trace variables and flux variables on the mesh skeleton at the element interfaces. We will discuss this aspect thoroughly in Section 4 when proposing the DPG method in the Ultraweak and primal formulations for the option pricing problem.

3. Functional Spaces and Preliminaries

We define the following energy spaces to handle the option pricing problem:

$$\begin{aligned} L^2(\Omega) &= \{f : \Omega \to \mathbb{R} \mid \quad \int_\Omega |f|^2 dx \leq \infty\}, \\ H^1(\Omega) &= \{f : \Omega \to \mathbb{R} \mid \quad f \in L^2(\Omega),\ f' \in L^2(\Omega)\}, \end{aligned} \tag{13}$$

where the L_2-norm is defined as

$$\|f\| := (f, f)^{\frac{1}{2}} = \left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}}. \tag{14}$$

The domain of the problem, denoted by Ω, is partitioned into a set of computational domains Ω_h consisting of open, disjoint elements $\gamma_h \in \Omega_h$. With the finite element mesh Ω_h, we can define the corresponding *broken* energy spaces as

$$\begin{aligned} L^2(\Omega_h) &= \{f \in L^2(\Omega) \mid \quad f \in L^2(\gamma_h),\ \forall \gamma_h \in \Omega_h\}, \\ H^1(\Omega_h) &= \{f \in L^2(\Omega) \mid \quad f \in H^1(\gamma_h),\ \forall \gamma_h \in \Omega_h\}. \end{aligned} \tag{15}$$

When using the broken test space, we also need to define the energy space for the trace variable. We define these spaces on the mesh skeleton Γ_h as follows:

$$H^{\frac{1}{2}}(\Gamma_h) = \{\hat{f} \in \prod_{\gamma_h \in \Omega_h} L^2(\partial \gamma_h) \mid \exists y \in H^1(\Omega) \text{ s.t. } \phi(y\big|_{\gamma_h}) = \hat{f}\}, \tag{16}$$

where the operator $\phi(\cdot)$ is the continuous trace operator, which can be defined element-wise as

$$\phi : H^1(\Omega_h) \to \prod_{\gamma_h \in \Omega_h} L_2(\partial \gamma_h). \tag{17}$$

Moreover, we need to define an appropriate space for the variational inequality defined in the problem of American option pricing. Thus, we define a half space $\mathcal{H}$ as follows:

$$\mathcal{H} := \{f \in L_2(\mathbb{R}^+) | \quad f \geq \bar{f}\}, \tag{18}$$

where $\bar{f} \in L^2(\mathbb{R}^+)$ is the obstacle function. More details about this space can be found in Achdou and Pironneau (2005), Trémolières *et al.* (2011). It is worth noting that for option pricing in one dimension, we consider a uniform discretization of the time interval $[0, T]$ and a truncated domain of space $[x_{\min}, x_{\max}]$ as the finite element mesh Ω_h, where the computational domain for all problems is $[-6, 6]$, except for the Asian option, which is $[-2, 2]$.

4. Pricing Vanilla Options

In this section, we apply the DPG method introduced in section 2 to numerically solve the option pricing problem, specifically focusing on pricing vanilla European options based on the Black-Scholes model.

4.1. *Vanilla European Options Based on the Black-Scholes Model*

In this part, we utilize the DPG method to solve the popular Black-Scholes Model, which provides a closed-form solution for all European-type derivatives, also known as vanilla options. Although the assumptions of this model are not universally valid, there is still a large group of market participants who utilize the Black-Scholes model with a premium, Higham (2004). Moreover, this model can serve as a benchmark to assess the efficiency of the DPG method.

Let's briefly recall the Black-Scholes model. The model assumes that the price of a risky asset, denoted as S_t, evolves according to the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{19}$$

where W_t is the Wiener process defined on an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}_t)$, r is the risk-free interest rate, and σ is the volatility of the return on the underlying security. This stochastic differential equation is commonly referred to as geometric Brownian motion.

Consider a European-style call option on an underlying asset S_t, where the spot price S_t satisfies the geometric Brownian motion equation (19), and the payoff at the expiration date T for a striking price K is given by $\max\{S_T - K, 0\} = (S_T - K)_+$. We are interested in determining the fair price of this option at the current time t, denoted as $U(S_0, 0)$, where $U(S_t, t)$ represents the value of the option when the underlying price is S_t. The Black-Scholes formula expresses the option value as:

$$U(S_t, t) = \mathbb{E}^Q \left(e^{-\int_t^T r_s ds} (S_T - K)_+ | \mathcal{F}_t \right), \tag{20}$$

where Q is the risk-neutral probability measure.

It can be shown Achdou and Pironneau (2005), Higham (2004) that the option price $U(S_t, t)$ satisfies the following deterministic partial differential equation:

$$\frac{\partial U}{\partial t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} - rU(S, t) = 0, \tag{21}$$

with the following boundary conditions:

$$\begin{aligned} &U(0, t) = 0, \quad \forall t \in [0, T], \\ &\lim_{S_t \to \infty} U(S_t, t) = S_t - e^{-r(T-t)}, \quad \forall t \in [0, T]. \end{aligned} \tag{22}$$

By exploiting the upper tail of the standard normal distribution:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz, \tag{23}$$

and defining

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}, \tag{24}$$

the analytical solution of Eq. (21) for a European call option is given by:

$$U(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2). \tag{25}$$

We use the closed-form analytical solution (25) for the European call option as a benchmark to study the accuracy and efficiency of the DPG method. By switching to log-prices $x = \log\left(\frac{S_t}{S_0}\right)$ and introducing the variable $\tau = T - t$, the partial differential Eq. (21) and boundary conditions (22) can be transformed into the following initial value constant coefficient partial differential equation:

$$\begin{cases} \dfrac{\partial U}{\partial \tau} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 U}{\partial x^2} - (r + \dfrac{\sigma^2}{2})\dfrac{\partial U}{\partial x} + rU(x,\tau) = 0, \\ \\ U(x,0) = \max\{e^x - K, 0\}, \\ \\ U(0,\tau) = 0, \end{cases} \tag{26}$$

Note that Eq. (26) can be used to price derivatives whose payoff depends on the price of the underlying asset at the maturity date. More complicated options with path-dependent prices, such as American options and Asian options, require different approaches, which we will discuss in the upcoming sections.

For the time discretization of problem (26), we employ the finite difference θ-method, which takes the following form:

$$\frac{u^{n+1} - u^n}{\Delta\tau} - (\theta\mathcal{L}_{BS}u^{n+1} + (1-\theta)\mathcal{L}_{BS}u^n) = 0, \tag{27}$$

for $n = 0, 1, 2, \ldots, N_\tau - 1$, with the time step $\Delta\tau = T/N_\tau$ and the implicitness factor $\theta \in [0,1]$. The operator $\mathcal{L}BS$ is defined as follows:

$$\mathcal{L}_{BS}u = \frac{\sigma^2}{2}\frac{\partial^2 u(x,\tau)}{\partial x^2} - (r + \frac{\sigma^2}{2})\frac{\partial u(x,\tau)}{\partial x} + ru(x,\tau),$$

Thus, different values of θ lead to different well–known timei–stepping schemes, such as the Backward Euler method ($\theta = 1.0$), Crank-Nicolson method ($\theta = 0.5$), and Forward Euler method ($\theta = 0.0$). The numerical efficiency of the finite difference method is well–established in the literature Bulirsch *et al.* (2002).

Next, we proceed to introduce the DPG methodology for the spatial discretization of the problem. Various variational formulations can be developed for the semi-discrete model problem (27) with different properties. In this study, we focus on two formulations: the classical (primal) formulation and the ultraweak formulation.

4.2. *Primal formulation for Vanilla options*

In this subsection, we propose the standard classical variational formulation for the DPG method, known as the DPG primal formulation. By testing the semi-discrete problem (27) with an appropriate test function v, integrating over the domain, and applying Green's identity, we obtain the following equation:

$$
\begin{aligned}
&(u^{n+1},v)_{\Omega_h}-(u^n,v)_{\Omega_h}\\
&\quad+\Delta\tau\theta\Big[-\Big(\frac{\sigma^2}{2}\frac{\partial}{\partial x}u^{n+1},\frac{\partial}{\partial x}v\Big)_{\Omega_h}+\Big((r+\frac{\sigma^2}{2})\frac{\partial}{\partial x}u^{n+1},v\Big)_{\Omega_h}-(ru^{n+1},v)_{\Omega_h}+\langle\frac{\partial}{\partial x}u^{n+1},v\rangle_{\partial\Omega_h}\Big]\\
&\quad+\Delta\tau(1-\theta)\Big[-\Big(\frac{\sigma^2}{2}\frac{\partial}{\partial x}u^{n},v\Big)_{\Omega_h}+\Big((r+\frac{\sigma^2}{2})\frac{\partial}{\partial x}u^{n},v\Big)_{\Omega_h}-(ru^{n},v)_{\Omega_h}+\langle\frac{\partial}{\partial x}u^{n+1},v\rangle_{\partial\Omega_h}\Big]=0,
\end{aligned}
\tag{28}
$$

where $(\cdot,\cdot)$ represents the standard inner product in the Hilbert space L_2, and $\langle\cdot,\cdot\rangle$ denotes the duality pair in $L^2(\Gamma)$. In the DPG methodology, the trial space is tested with a broader discontinuous (broken) space, and thus we do not assume that the test functions vanish on the Dirichlet boundary conditions. However, the term $\frac{\partial u^n}{\partial x}$ is recognized as the flux variable $\hat{q}^n$, which is a new unknown on the mesh skeleton. Therefore, by defining a new group variable $\mathbf{u}=(u,\hat{q})\in H^1(\Omega)\times H^{-1/2}(\partial\Omega)$, the broken primal formulation for the Black-Scholes Eq. (26) can be expressed as:

$$
\begin{cases}
b_{\text{primal}}(\mathbf{u},v)=l(v),\\
\mathbf{u}(e^x,0)=\max\{x-K,0\},\\
\mathbf{u}(0,\tau)=0,
\end{cases}
\tag{29}
$$

where the bilinear form $b_{\text{primal}}(\cdot,\cdot)$ and the linear functional $l(\cdot)$ are defined as follows:

$$
\begin{aligned}
b_{\text{primal}}(\mathbf{u},v)&=(u^{n+1},v)_{\Omega_h}+\Delta\tau\theta\Big[-\Big(\frac{\sigma^2}{2}\frac{\partial}{\partial x}u^{n+1},\frac{\partial}{\partial x}v\Big)_{\Omega_h}+\Big((r+\frac{\sigma^2}{2})\frac{\partial}{\partial x}u^{n+1},v\Big)_{\Omega_h}\\
&\quad-(u^{n+1},v)_{\Omega_h}+\langle\hat{q}^{n+1},v\rangle_{\partial\Omega_h}\Big],\qquad n=1,\cdots,N_t,\\
l(v)&=(u^n,v)_{\Omega_h}+\Delta\tau(1-\theta)\Big[\Big(\frac{\sigma^2}{2}\frac{\partial}{\partial x}u^{n},\frac{\partial}{\partial x}v\Big)_{\Omega_h}+\Big((r+\frac{\sigma^2}{2})\frac{\partial}{\partial x}u^{n},v\Big)_{\Omega_h}\\
&\quad-(u^{n},v)_{\Omega_h}-\langle\hat{q}^{n},v\rangle_{\partial\Omega_h}\Big],\qquad n=1,\cdots,N_t,
\end{aligned}
\tag{30}
$$

with the boundary conditions $\mathbf{u}^0=\max\{e^x-K,0\}$ for all $x\in\Omega_h$, and $\mathbf{u}^i(x=0)=0$ for $i=1,\cdots,N_t$. Here, element-wise operations are denoted by the subscript h. Having the new flux unknown on the mesh skeleton in the primal formulation (30) is the price we pay for using a larger test space (enriched test space).

4.3. *Ultraweak Formulation for Vanilla Options*

In this section, we will derive the ultraweak DPG formulation for the pricing problem. The first step in obtaining the ultraweak formulation is to transform the Black-Scholes problem (26) into a first-order system of differential equations by introducing a new variable $\vartheta(x,t)=\frac{\partial U}{\partial x}(x,t)$ for

$(x,t) \in \Omega \times [0,T]$. The transformed system is given by:

$$\begin{cases} \dfrac{\partial U}{\partial \tau} - \dfrac{\sigma^2}{2}\dfrac{\partial \vartheta}{\partial x} - (r + \dfrac{\sigma^2}{2})\vartheta + rU(x,t) = 0, \\ \\ \vartheta - \dfrac{\partial U}{\partial x} = 0, \\ \\ U(x,0) = \max\{e^x - K, 0\}, \\ U(0,\tau) = 0. \end{cases} \tag{31}$$

Next, by defining a new group variable $\mathbf{u} = (u, \vartheta)$ and testing Eq. (31) with the test variables $\mathbf{v} = (v, \omega)$, we can integrate and use Green's identity to obtain the following ultraweak formulation:

$$\begin{aligned} &(u^{n+1}, v)_{\Omega_h} + (u^n, v)_{\Omega_h} + \\ &\Delta\tau\theta\Big[(\vartheta^{n+1}, \tfrac{\sigma^2}{2}\tfrac{\partial}{\partial x}v)_{\Omega_h} + (\vartheta^{n+1}, (r + \tfrac{\sigma^2}{2})v)_{\Omega_h} - (u^{n+1}, v)_{\Omega_h} + (u^{n+1}, \tfrac{\partial}{\partial x}\omega)_{\Omega_h} - (\vartheta^{n+1}, \omega)_{\Omega_h} + \\ &\langle \tfrac{\partial}{\partial x}u^{n+1}, v\rangle_{\partial\Omega_h} + \langle \tfrac{\partial}{\partial x}\vartheta^{n+1}, v\rangle_{\partial\Omega_h}\Big] + \Delta\tau(1-\theta)\Big[(\vartheta^n, \tfrac{\sigma^2}{2}\tfrac{\partial}{\partial x}v)_{\Omega_h} + (\vartheta^n, (r + \tfrac{\sigma^2}{2})v)_{\Omega_h} - \\ &(u^n, v)_{\Omega_h} - (u^n, \tfrac{\partial}{\partial x}\omega)_{\Omega_h} - (\vartheta^n, \omega)_{\Omega_h} + \langle \tfrac{\partial}{\partial x}u^n, v\rangle_{\partial\Omega_h} + \langle \tfrac{\partial}{\partial x}\vartheta^n, v\rangle_{\partial\Omega_h}\Big] = 0, \end{aligned} \tag{32}$$

It is important to note that in the ultraweak formulation, a discontinuous test space is utilized, which conforms to the DPG methodology. Additionally, the weak formulation does not include derivatives of the trial variable, and these trial variables are defined in $L_2(\Omega)$. Consequently, the boundary values of the field variables are irrelevant on the skeleton Γ. To address this, we introduce two trace variables, $\hat{u}_{n+1} \in H^{1/2}(\Omega)$ and $\hat{\vartheta}^{n+1} \in H^{1/2}(\Omega)$, which are unknown on the skeleton.

By defining the group variables $\mathbf{u} = (u, \vartheta)$, $\hat{\mathbf{u}} = (\hat{u}, \hat{\vartheta})$, and $\mathbf{v} = (v, \omega)$, the broken ultraweak formulation for the Black-Scholes model is finding $\mathbf{u} = (u, \vartheta) \in L_2(\Omega) \times L_2(\Omega)$ and $\hat{\mathbf{u}} = (\hat{u}, \hat{\vartheta}) \in H^{1/2}(\Omega) \times H^{1/2}(\Omega)$ such that:

$$\begin{cases} b_{\text{ultraweak}}((\mathbf{u}, \hat{\mathbf{u}}), \mathbf{v}) = l(\mathbf{v}) \\ \\ (\mathbf{u}, \hat{\mathbf{u}})|_{(x,0)} = \max\{e^x - K, 0\}, \\ \\ (\mathbf{u}, \hat{\mathbf{u}})|_{(0,\tau)} = 0, \end{cases} \tag{33}$$

where

$$
\begin{aligned}
b_{\text{ultraweak}}((\mathbf{u},\hat{\mathbf{u}}),\mathbf{v}) &= b_{\text{ultraweak}}(((u,\vartheta),(\hat{u},\hat{\vartheta})),(v,\omega)) \\
&= (u^{n+1},v)_{\Omega_h} + \Delta\tau\theta\Big[(\vartheta^{n+1},\frac{\sigma^2}{2}\frac{\partial}{\partial x}v)_{\Omega_h} + (\vartheta^{n+1},(r+\frac{\sigma^2}{2})v)_{\Omega_h} - (u^{n+1},v)_{\Omega_h} - \\
&(u^{n+1},\frac{\partial}{\partial x}\omega)_{\Omega_h} - (\vartheta^{n+1},\omega)_{\Omega_h} + \langle \hat{u}^{n+1},v\rangle_{\partial\Omega_h} + \langle \hat{\vartheta}^{n+1},v\rangle_{\partial\Omega_h}\Big], \qquad n=1,\cdots,N_t. \\
l(\mathbf{v}) = l(v,\omega) &= (u^n,v)_{\Omega_h} + \Delta\tau(1-\theta)\Big[(\vartheta^n,\frac{\sigma^2}{2}\frac{\partial}{\partial x}v)_{\Omega_h} + (\vartheta^n,(r+\frac{\sigma^2}{2})v)_{\Omega_h} - (u^n,v)_{\Omega_h} + \\
&- (u^n,\frac{\partial}{\partial x}\omega)_{\Omega_h} - (\vartheta^n,\omega)_{\Omega_h} + \langle \hat{u}^n,v\rangle_{\partial\Omega_h} + \langle \hat{\vartheta}^n,v\rangle_{\partial\Omega_h}\Big], \qquad n=1,\cdots,N_t,
\end{aligned} \tag{34}
$$

4.4. *Solvibility of the Primal and Ultraweak Variational formulations*

Here, we use a discontinuous test space that is conforming on an element-wise basis, as per the DPG methodology. Additionally, in the derived variational formulations, the choice of the test space's inner product significantly impacts the DPG method, as it determines the norm and structure of the test space where the DPG method becomes optimal. For example, if the errors in L_2-norm are of interest, the graph norm is a suitable choice for the test space in the ultraweak formulation, and the standard energy norm induced by the bilinear form $\|\cdot\|_E = b_{\text{primal}}(v,v)$ is suitable for the primal formulation. In this paper, we propose the following graph norm for primal formulation (29), and the ultraweak formulation (33)

$$
\begin{aligned}
&\text{Primal}: \|v\|_V^2 = \frac{1}{\Delta t}\|v\|^2 + \frac{1}{(\Delta t)^2}\|\frac{\sigma^2}{2}\frac{\partial}{\partial x}v\|^2, \\
&\text{Ultraweak}: \|\mathbf{v}\|_V^2 = \|(v,\omega)\|_V^2 = \frac{1}{(\Delta t)^2}\|\frac{\sigma^2}{2}\frac{\partial}{\partial x}v - rv - \omega\|^2 + \frac{1}{\Delta t}\|(r+\frac{\sigma^2}{2})v - \frac{\partial}{\partial x}\omega\|^2,
\end{aligned} \tag{35}
$$

Having defined the graph norm and energy norm in (35), and the inner product of the corresponding test space as a result of this choice, we are now prepared to discretize the weak forms and construct the DPG system. In classical Galerkin methods, the convention is to choose the same discrete space for both the trial and test spaces, resulting in a square linear system. However, in the DPG method, the discrete trial space $U_h \subset U$ and test space $V_h \subset V$ have different dimensions. In practical DPG methods with optimal test spaces, the test space is enriched, such that $\dim V_h \geq \dim U_h$. We assume that $\{u_j\}_{j=1}^N$ and $\{v_j\}_{j=1}^M$ are the bases of the trial and test spaces, respectively, where $M \geq N$. In the DPG methodology, each trial space basis function u_i and its corresponding optimal test function v_i^{opt} satisfy the following system:

$$
(v_i^{\text{opt}},\delta v)_V = b(u_i,\delta v), \quad \forall \delta v \in V. \tag{36}
$$

Now, let's define the $M \times M$ Gram matrix $G = (G_{ij})_{M\times M}$ as

$$G_{ij} = (v_i, v_j)_V,$$

and the $N \times M$ stiffness matrix $B = (B_{ij})_{N\times M}$ as

$$B_{ij} = b(u_i, v_j),$$

For the primal formulation, finding the matrix B is straightforward from the bilinear form and test norm. However, calculating this matrix for the ultraweak formulation can be more involved. The stiffness matrix B has the following structure:

$$B = \begin{bmatrix} B_{uv} & B_{\vartheta v} & B_{\hat{u}v} & B_{\hat{\vartheta}v} \\ B_{u\omega} & B_{\vartheta\omega} & B_{\hat{u}\omega} & B_{\hat{\vartheta}\omega} \end{bmatrix}_{N\times M}, \tag{37}$$

and l represents the mass matrix $l(v) = (f, r)$. We use high-order Lagrange basis functions of different orders to expand the trial space with order P, and we enrich the test space with order $p + \Delta p$ for $\Delta p = 2$. Thus, the global assembly takes the following form:

$$B^{\text{n-op}}\mathbf{u}_h = B^T G^{-1} B \mathbf{u}_h = B^T G^{-1} l = l^{\text{n-op}}, \tag{38}$$

Here, the discrete operators $B^{\text{n-op}}$ and $l^{\text{n-op}}$ represent the near-optimal mass and stiffness matrices for the DPG formulation. It is worth noting that, thanks to the broken structure of the test space, the evaluation of optimal test functions in the Gram matrix and its inversion can be localized, allowing for efficient parallelization of the global assembly. This characteristic makes the DPG method practical for solving the option pricing problem.

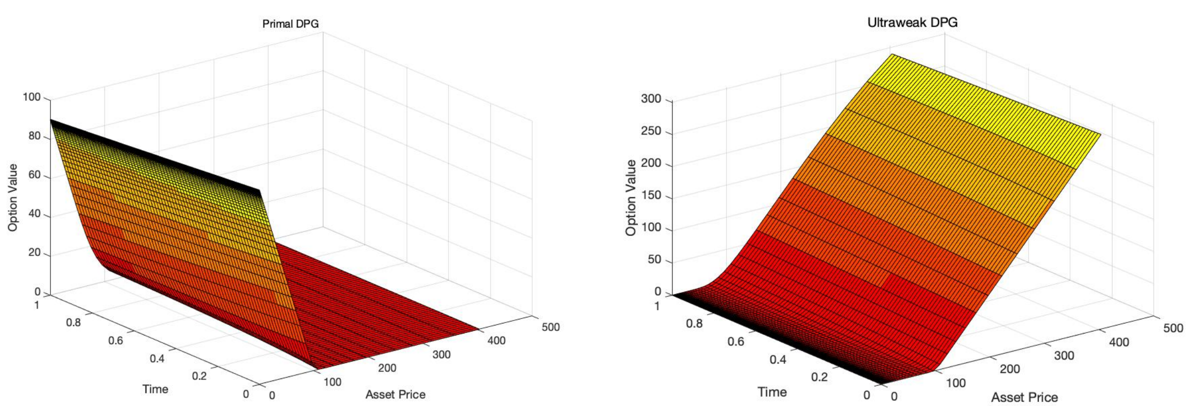


Figure 1. European put

Figure 2. European call

Figure 3. The surface for of two European options using DPG method with $\sigma = 0.4$, $r = 0.1$, and $k = 100$.

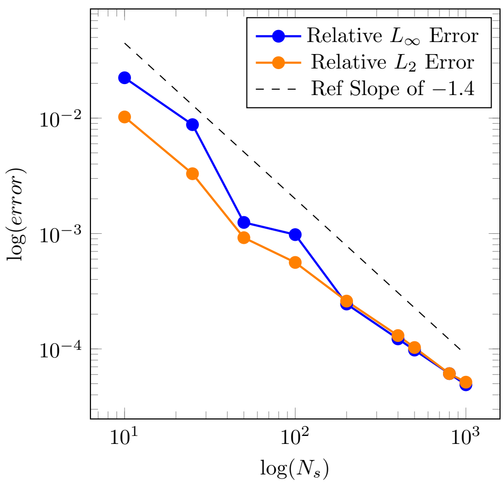
4.5. *Numerical Results*

In this section, we provide numerical experiments to showcase the efficiency and accuracy of the DPG method in pricing vanilla options using both the primal and ultraweak DPG methods. For this experiment, risk-free rate r is set to be 0.05, time to maturity T is one year, and the strike price K is 100. The computational domain is $[-6, 6]$, and a variety of values for the market volatility σ

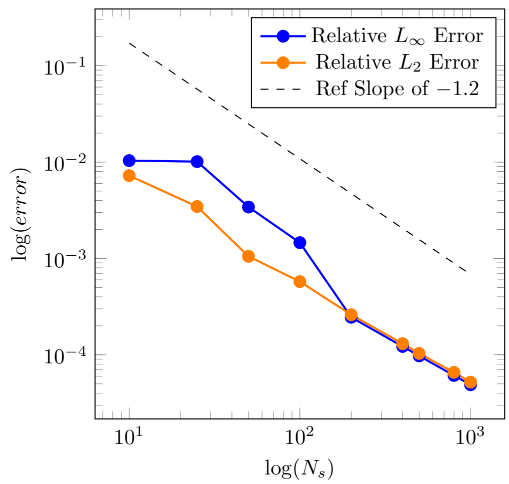
is considered in this part. Through this paper, we report the relative errors of L_2-error, L_∞-error of the solution obtained by the proposed numerical scheme. The binomial method implemented in Higham (2002) is utilized as a benchmark and analytical solution to compare with the approximated solution obtained with the DPG method. The relative errors are defined as follows

$$\|E\|_{L_2}^2 = \|\frac{u - \tilde{u}}{u}\|_{L_2}^2, \qquad \|E\|_\infty = \|\frac{u - \tilde{u}}{u}\|_{L_\infty}, \tag{39}$$

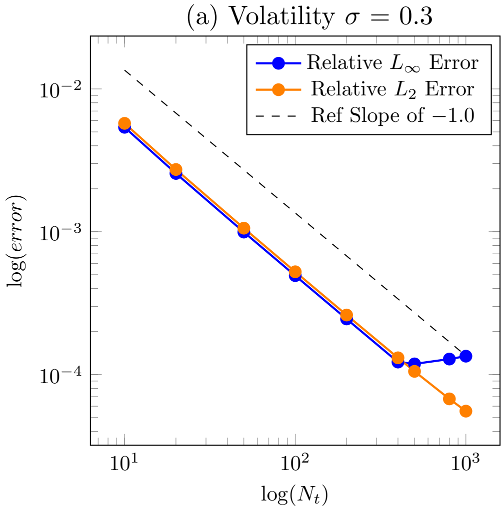
where $\tilde{u}$ represents the estimated value attained from the numerical method. Fig. 3 depicts the surface of a call and a put option with volatility $\sigma = 0.4$ for both primal and ultraweak DPG formulation. In this part of the experiment, we study the asymptotic convergence of relative errors



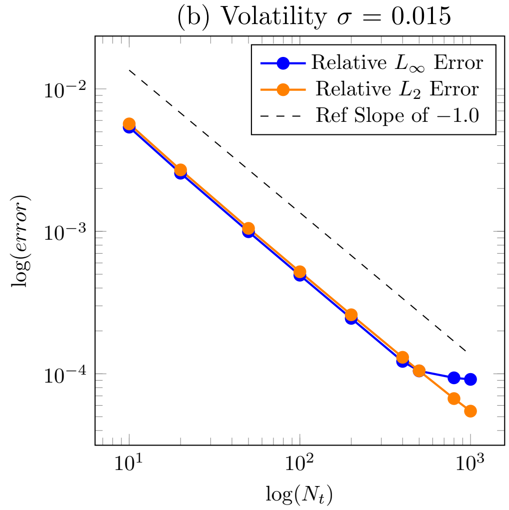
(a) Volatility $\sigma = 0.3$



(b) Volatility $\sigma = 0.015$



(c) Volatility $\sigma = 0.3$



(d) Volatility $\sigma = 0.015$

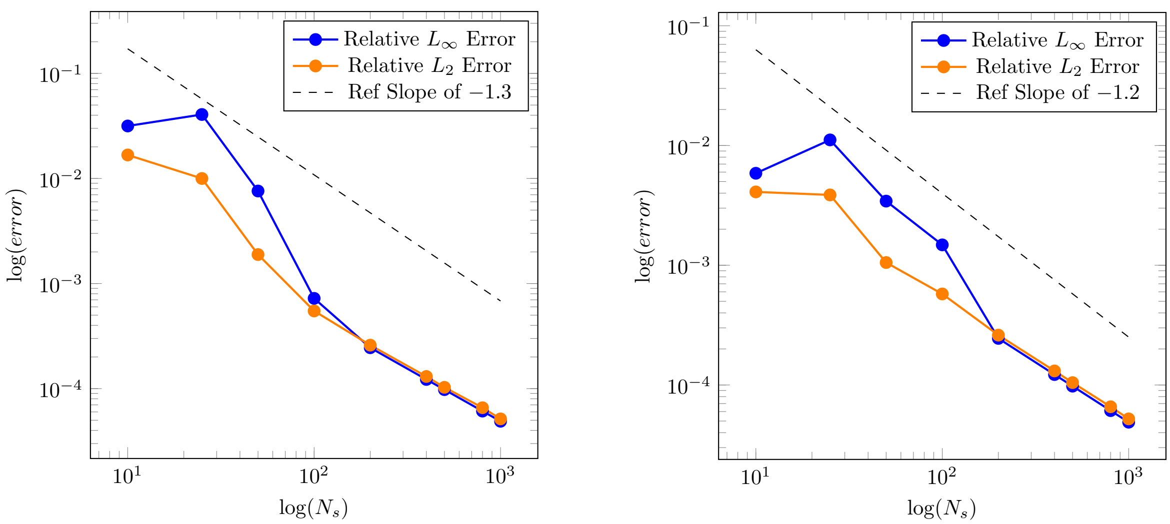
Figure 4. Accuracy properties of primal DPG for European put options $r = 0.05$, $K = 100$, and different volatility

of the numerical method for uniform mesh refinement both in time and steps. It is worth mentioning that error is small in general, and the relative error is of order of 10^{-6}.

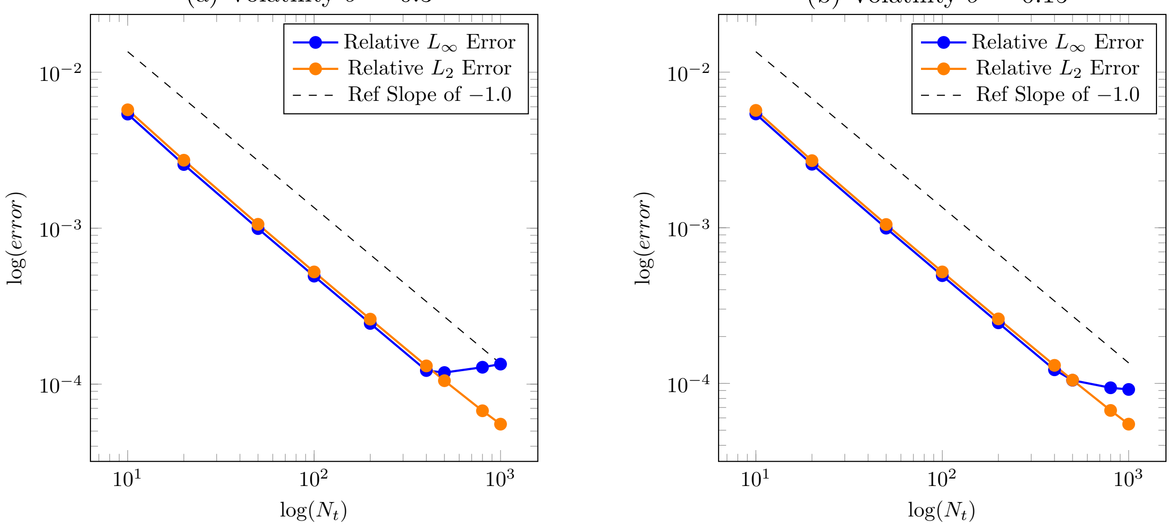
In this regard, Fig. (4a), and (4b) displays the space order of convergence of the primal DPG method for volatilises of $\sigma = 0.3$ and $\sigma = 0.015$ pricing a European put option. It is evident that the convergence rate of primal DPG scheme is super linear in space.

The same investigation for ultraweak DPG scheme Fig. (5a), and (5a) shows that although the convergence rate in space is super-linear the errors in this scheme decay moderately gently.

We observe that for the space order in both ultraweak and primal schemes initially we see some inconsistency in the linear decreasing of the error but once a number of elements approach a certain point, we witness the expected linear convergence $\mathcal{O}(h)$, which can cause this overall super-linear convergence rate. However, Fig. (4c), and (4d), and Fig. (5c), and Fig. (5d) depicts this



(a) Volatility $\sigma = 0.3$ (b) Volatility $\sigma = 0.15$



(c) Volatility $\sigma = 0.3$ (d) Volatility $\sigma = 0.015$

Figure 5. Accuracy properties of ultraweak DPG for European put options r=0.05, K=100, and different volatility

observation more precisely when for the same scenario the rate of convergence for the Primal DPG and Ultraweak DPG method is linear in time due to the fact that the $h = 0.01$ is fixed for this experiment.

5. Exotic Options

Financial institutions issue other forms of options that are not vanilla call or put introduced in section 4. This modern financial instrument is traded between companies and banks to cope with a variety of demands Zhu *et al.* (2004). So, exotic options are traded in the over-the-counter (OTC) market to satisfy special needs. Being a complicated financial instrument is the common property of exotic options where the value of of these options depends on the whole or part of the path of the underlying security. Thus, exotic options are path-dependent options. In this section, we proposed the DPG method for the numerical solution of the important examples of path-dependent exotic options including American options, Asian options, Barrier options, and look-back options.

5.1. *American options*

In this section, we briefly review American option pricing under the simple model of Black-Scholes. Contrary to the European option, the holder of this contract has the right to exercise the option at any time before maturity. It is well known that this slight difference brings the analysis of American options much more complicated. Indeed, this right turn problem of valuing the American option into a stochastic optimization problem. The price of an American option under the risk-neutral pricing principle can be obtained as

$$U(x,t) = \sup_{t\leq\tau\leq T} \mathbb{E}[e^{-\int_t^\tau r(s)ds} h(x)|\mathcal{F}_t], \tag{40}$$

where $h(x)$ is the option payoff, and τ is a stopping time. Stopping time is the time that owner of the option exercises the contract, besides, the stopping time is a concept in the stochastic analysis as well Chung (2013). It is worth noting that due to the complexity of the American option problem, this problem does not have a closed-form solution. One way of formulating American options thanks to the no-arbitrage principle is the free boundary value problem. Indeed, the free boundary happens when the option is deep in-the-money, and finding this boundary alongside pricing the American option brings extra difficulties to the problem. Here we briefly recall the different forms of American options and the corresponding DPG formulation for the formulations, for more detail one can see Seydel and Seydel (2006).

Considering the log-prices $x = \log(\frac{S_t}{S_0})$, changing tenor $T-t$ to τ, the free boundary formulation of the American put option yields:

$$\begin{cases} \dfrac{\partial U}{\partial \tau}(x,\tau) - \dfrac{\sigma^2}{2}\dfrac{\partial U^2}{\partial x^2}(x,\tau) - (r+\dfrac{\sigma^2}{2})\dfrac{\partial U}{\partial x}(x,\tau) + rU(x,\tau) = 0, & \forall x > S_f, \\ U(x,\tau) = K - e^x, \qquad \forall x \leq S_f, \\ U(x,0) = (K-e^x)^+, \\ \lim\limits_{x\to\infty} U(x,\tau) = 0, \\ \lim\limits_{x\to S_f} U(x,\tau) = K - e^{S_f}, \\ \lim\limits_{x\to S_f} \dfrac{\partial U(x,\tau)}{\partial x} = -1, \end{cases} \tag{41}$$

in which, S_f is the free boundary of the American option pricing. It is evident that solving the problem of American option in a free boundary framework needs evaluating the free boundary along the finding the solution. Whereas, There is another approach to deriving the American option pricing problem called a linear complementarity problem (LCP). The advantage of this approach is that free boundary is not present in the formulation anymore. However, solving the LCP problem has its own complexity, and techniques Murty and Yu (1988) . The linear complementarity problem

(LCP) of the American option can be written as

$$\begin{cases} \left(\frac{\partial U}{\partial \tau}(x,\tau) - \frac{\sigma^2}{2}\frac{\partial U^2}{\partial x^2}(x,\tau) - (r+\frac{\sigma^2}{2})\frac{\partial U}{\partial x}(x,\tau) + rU(x,\tau)\right)(U(x,\tau)-h(x)) = 0, \\ \\ \frac{\partial U}{\partial \tau}(x,\tau) - \frac{\sigma^2}{2}\frac{\partial U^2}{\partial x^2}(x,\tau) - (r+\frac{\sigma^2}{2})\frac{\partial U}{\partial x}(x,\tau) + rU(x,\tau) \geq 0, \\ \\ U(x,\tau) - h(x) \geq 0, \\ U(x,0) = (K-e^x)^+. \end{cases} \tag{42}$$

The main approach here is to utilize the DPG formulation for the governing equations of the Eq. (42), and (41) and then consider the free boundary condition for them. The using DPG method for a (LCP) is examined before in Führer *et al.* (2018) for using DPG formulation for the Signorini type problem as a contact problem. However, Thomas Fuhrer et al. in Führer *et al.* (2018) proposed the ultraweak formulation of the corresponding problem, here we derive both ultraweak and primal formulation of the DPG method for the problem of American option pricing as a special case of obstacle problem.

Now, for the DPG formulation in LCP framework, we multiply the second inequality condition in the Eq. (42) with the smooth no-negative test functions $v \in V$ where test space is a broken convex cone and following the same process of defining trail and flux variable presented in the section 4, and after some integration by part we obtain

$$\frac{d}{d\tau}(\mathbf{u},\mathbf{v}) + b^{\tau}(\mathbf{u},\mathbf{v}) \geq 0, \tag{43}$$

where bilinear form for primal formulation defines as

$$b^{\tau}_{\text{primal}}(\mathbf{u},v) = (-\frac{\sigma^2}{2}\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x})_{\Omega_+} + \big((r+\frac{\sigma^2}{2})\frac{\partial u}{\partial x}, v\big)_{\Omega_+} - (u,v)_{\Omega_+} + \langle \hat{q}, v\rangle_{\partial\Omega_+}, \tag{44}$$

where Ω_+ shows the non–negative part of the domain, with a set of trial and flux variables $\mathbf{u} = (u,\hat{q}) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, and test variable $\mathbf{v} = v \in \mathcal{H}^1(\Omega)$. Moreover, defining trail variables $\mathbf{u} = (u,\vartheta) \in L_2(\Omega) \times L_2(\Omega)$, and flux variables $\hat{\mathbf{u}} = (\hat{u},\hat{\vartheta}) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, one can define the bilinear form in (43) for the ultraweak formulation as following

$$\begin{aligned} b^{\tau}_{\text{ultraweak}}((\mathbf{u},\hat{\mathbf{u}}),\mathbf{v}) = b^{\tau}_{\text{ultraweak}}\Big(((u,\vartheta),(\hat{u},\hat{\vartheta})),(v,\omega)\Big), \\ \\ = (\vartheta, \frac{\sigma^2}{2}\frac{\partial v}{\partial x})_{\Omega_+} + (\vartheta,(r+\frac{\sigma^2}{2})v)_{\Omega_+} - (u,v)_{\Omega_+} - (u,\frac{\partial \omega}{\partial x})_{\Omega_+} \\ - (\vartheta,\omega)_{\Omega_+} + \langle \hat{u}, v\rangle_{\partial\Omega_+} + \langle \hat{\vartheta}, v\rangle_{\partial\Omega_+}. \end{aligned} \tag{45}$$

It is well-known that the two variational inequalities proposed, derived from the bilinear forms obtained through equation (43) via (44) and (45), are the parabolic variational inequalities of the first kind that admit a unique solution Kinderlehrer and Stampacchia (2000). Having well–posed variational inequality of (43), we can approximate the problem in a finite-dimensional space. Thus, similar to estimating the price of vanilla options, we consider the time partition $\tau_1 = 0 \leq \cdots \leq \tau_{N_\tau} = T$ of the time interval $[0,T]$, and discrete trial space $U_h \subset U$, and enriched test space

$V_h \subset V$ ($\dim V_h \geq \dim U_h$) and the corresponding basis spanned $\{u_j\}_{j=1}^N$, and $\{v_j\}_{j=1}^M$ for the aforementioned spaces. We use the backward finite difference Euler method to approximate the time derivative, and as a result, the discrete DPG for variational inequalities arising from the American option pricing problem yields

$$(u^{n+1} - u^n, \mathbf{v}) + \Delta\tau b_n^\tau(u^n, \mathbf{v}) \geq 0, \qquad \forall \mathbf{v} \in V_h. \tag{46}$$

However, writing the θ-method for the second term in left hand side of the discrete variational inequality (46) will be performed very similarly to what is proposed for vanilla options. Let B and G be the stiffness and Gram matrices defined by

$$B_{ij} = b_n^\tau(u_i, \mathbf{v}_j), \qquad G_{ij} = (\mathbf{v}_i, \mathbf{v}_j)_V, \qquad l_i = (u_i, \mathbf{v}_j)_V, \tag{47}$$

where $(\cdot,\cdot)_v$ inner product of test space obtained from the energy norm for primal DPG and graph norm for ultraweak form introduced in (35). So, the discrete variational inequality (46) is equivalent to

$$\begin{cases} B^T G^{-1} l(u^{n+1} - u^n) + \Delta\tau B^T G^{-1} B u^n \geq 0, \\ u^n \geq h(x), \\ (u^n - h(x))\big(B^T G^{-1} l(u^{n+1} - u^n) + \Delta\tau B^T G^{-1} B u^n\big) = 0, \end{cases} \tag{48}$$

for $n = 1, \cdots, N_\tau$. Setting near the optimal discrete operators of $B^{\text{n-op}} = B^T G^{-1} B$, $l^{\text{n-op}} = B^T G^{-1} l$ discrete LCP (48) will attain the following form

$$\begin{cases} l^{\text{n-op}}(u^{n+1} - u^n) + \Delta\tau B^{\text{n-op}} u^n \geq 0, \\ u^n - h(x) \geq 0, & \forall n = 1, \cdots, N_\tau \\ (u^n - h(x))\big(l^{\text{n-op}}(u^{n+1} - u^n) + \Delta\tau B^{\text{n-op}} u^n\big) = 0. \end{cases} \tag{49}$$

There are different approaches to solve the discrete variational inequality (49) including fix-point approach, penalization method, iterative method to just name few Damircheli and Bhatia (2019). The DPG method, being a minimum residual method, always produces a symmetric positive definite stiffness matrix. This ensures the convergence of solutions obtained through iterative algorithms, such as the semismooth Newton's method employed in this study.

To close the section we will present the DPG formulation for solving the free boundary value problem (41). Similar to the procedure for governing equation of vanilla options, one can test the governing equation (41) with the appropriate test functions, and define the following system

$$\frac{d}{d\tau}(\mathbf{u}, \mathbf{v}) + b^\tau(\mathbf{u}, \mathbf{v}) = 0, \qquad \forall x > S_f, \tag{50}$$

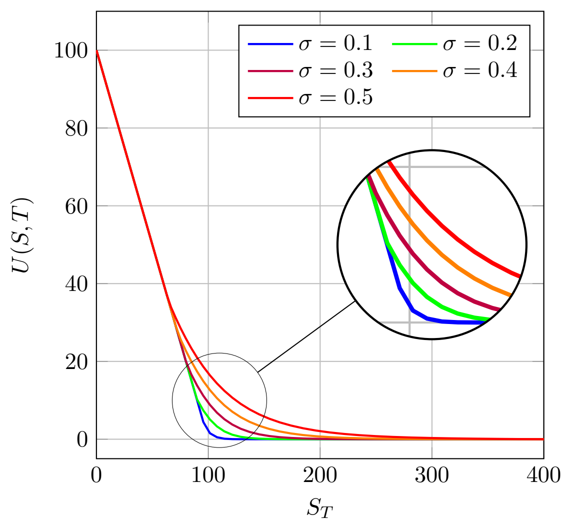
Where the *bilinear form* in the Eq. (50) has the form of (44) for the primal formulation and (45) for the ultraweak formulation. Like our approach so far, we use the Backward Euler method for time derivative and trial and test space defined for the LCP form to find the following discreet system of equation

$$(u^{n+1} - u^n, \mathbf{v}) + \Delta\tau b_n^\tau(u^n, \mathbf{v}) = 0, \quad \forall x_h > S_f, \qquad \forall \mathbf{v} \in V_h. \tag{51}$$

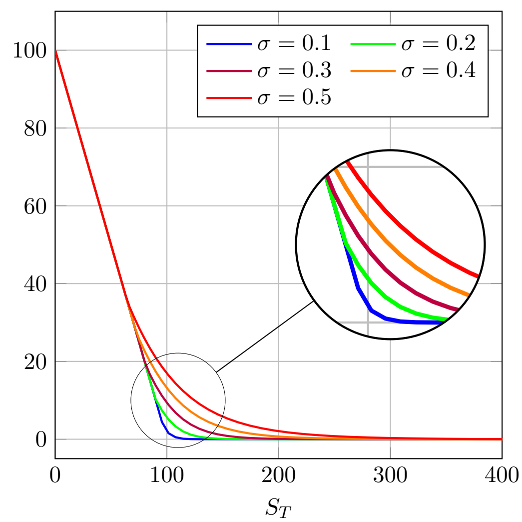
Having enough fine time discretization in the above form, using the information with one time step lag can attain a good approximation of the solution of the American option. In another word, one need to notice that the final price of the American option will find from the following implicitly boundary condition

$$u^n = \begin{cases} \max\{h(x), u^{n-1}\}, \forall x \in \Omega^o, \\ \\ h(x), & x = \inf \partial\Omega, \\ 0, & x = \sup \partial\Omega. \end{cases} \tag{52}$$

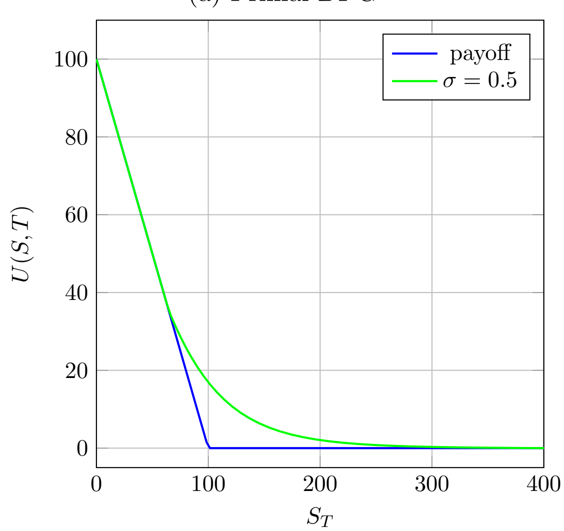
in which $h(x)$ is the payoff of American option. Boundary conditions presented in (52) are necessary boundary conditions of the problem of valuing American option pricing.



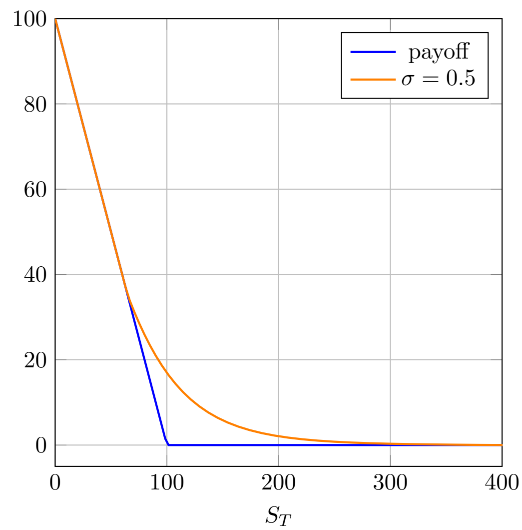
(a) Primal DPG



(b) Ultraweak DPG



(c) Primal DPG



(d) Ultraweak DPG

Figure 6. Value of American put option for r=0.05, K=100, and different volatilises.

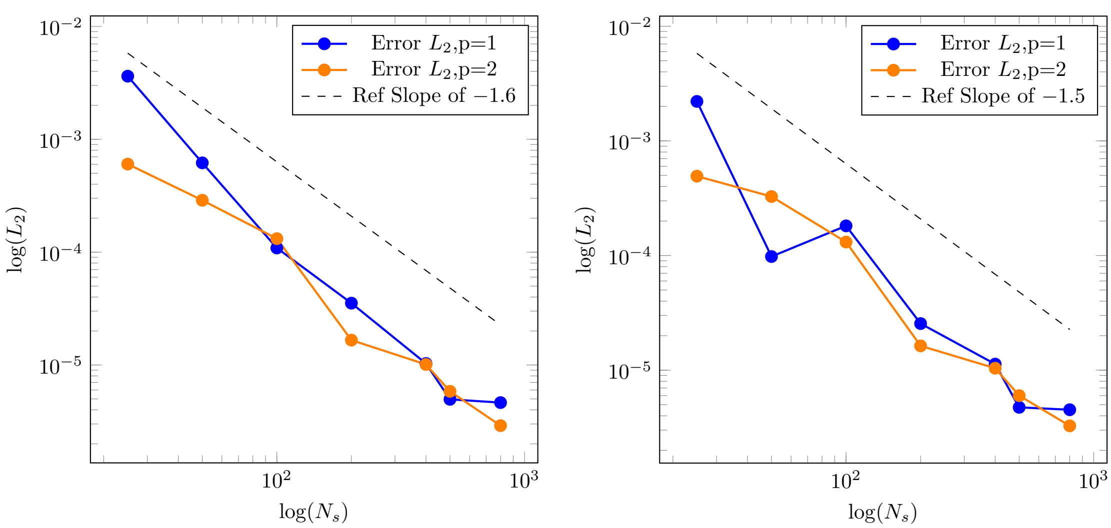
5.2. *Numerical Experiments*

In this set of numerical experiments, we study the problem of valuing the American option with the ultraweak and primal DPG method. We intend to verify that DPG is a reliable and efficient method for solving this free boundary value problem. Fig. (6a), and Fig. (6b) illustrate the price of an American put option for a fixed interest rate $r = 0.05$, maturity $K = 100$, and different volatilises. It is a well–known fact that the price of an American option is greater than a European option due to the right of the owner of the American option for exercising the financial contract anytime before maturity, this can vividly be seen in Fig. (6c), and Fig. (6d) for the payoff and value of an American option. Thus, the proposed methods can mimic this behavior accurately for different volatility of the market for both primal and ultraweak formulations. Error analysis of the

Table 1. Value of American Option r = 0.05, $\sigma = 0.15$, K=100

$\Delta\tau$	h	value		$\|E\|_\infty$	
		Primal	Ultraweak	Primal	Ultraweak
0.01	0.46	4.24860417	4.24224142	0.01599300	0.00963025
0.01	0.23	4.23640882	4.23311370	0.00379765	0.00050253
0.01	0.11	4.23335691	4.23394300	0.00074574	0.00133183
0.01	0.05	4.23295566	4.23288421	0.00034449	0.00027304
0.01	0.03	4.23255287	4.23254907	5.83E-05	6.21E-05
0.01	0.02	4.23256997	4.23256637	4.12E-05	4.48E-05
0.01	0.01	4.23259347	4.23259367	1.77E-05	1.75E-05

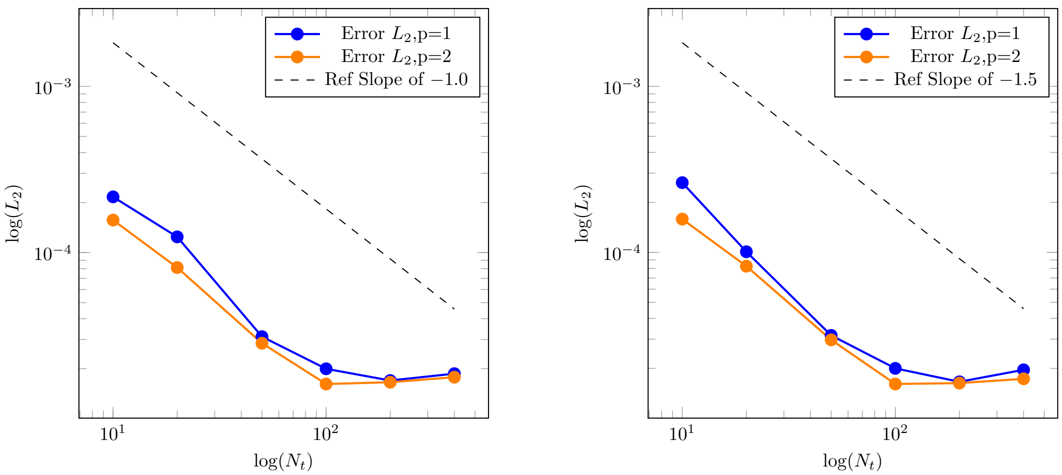
American option conducted with the relative L_∞, and L_2–error of the solution very similar to the definitions (39). Besides, the bench mark for the exact solution is opt the value of binomial method introduced and implemented in Higham (2002). Table (1) is prepared to show the error of the DPG numerical scheme for both primal and Ultraweak formulation. In this study, the time step is fixed $\Delta\tau = 0.01$, and we use a finer mesh in spatial dimension on each step. One can see that the trend of error is descending as h decreases and we get more accurate value of the American options. Although the magnitude of error is important, the order by which error is lessened is of a great



(a) primal DPG $\sigma = 0.15$ (b) ultraweak DPG $\sigma = 0.15$

Figure 7. Accuracy properties of ultraweak and primal DPG for American put options in the spatial dimension with the parameters $r = 0.05$, $K = 100$, and $\sigma = 0.15$

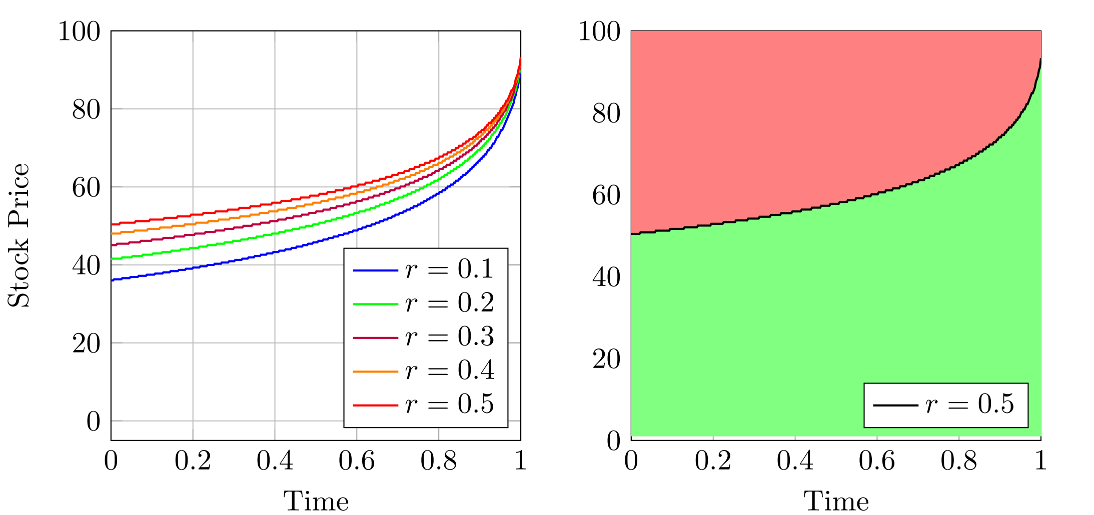
importance in our error analysis. In this investigation we used the high order DPG method as well to study the effect of the order of interpolation on the valuing of the American option pricing. Let's commence with the spatial order of convergency. Fig. (7a), and Fig. (7b) illustrates the order of convergence of both primal and ultraweak formulation for valuing American option for the fixed interest rate $r = 0.05$, exercise prices of $K = 100$, and the market volatility of $\sigma = 0.15$ in space order for first order and second order DPG. The experiment shows that asymptotic convergence of L_2–error is superlinear, but it doesn't reach the $O(h^2)$ for the second order DPG scheme. One possible explanation of the diminishing the order could be an adverse impact of free boundary in the pricing problem. However, the error is relatively small, and table (1) reinforce this trend as well for relative sup-error for both primal and ultraweak formulation, where ultraweak formulation has a tiny better performance in majority of cases. In order to study the stability and convergence



(a) primal DPG $\sigma = 0.15$ (b) ultraweak DPG $\sigma = 0.15$

Figure 8. Accuracy properties of Ultreawek and primal DPG scheme for American put options with respect to time step with parameters r=0.05, K=100, and $\sigma = 0.15$

in time stepping scheme, we prepared Fig. (8). A fixed mesh in space with $N_s = 64$ elements is used and decrease the time step $\Delta\tau$ and record the L_2-error for first and second order DPG method. The convergence analysis shows that this both primal Fig. (8a) and ultraweak Fig. (8b) formulation benefit from the rate of convergence of $\mathcal{O}(\Delta\tau)$ as we expected and the backward Euler method is unconditionally stable. However, the rate of convergence for time stepping captures for initial time steps (almost $N_\tau = 100$), where as after this point spacial discretization dictates it's impact afterwards for both DPG forms. Besides accurately pricing the American-type financial



(a) Optimal exercise boundaries (b) "Exercise" and "Do not Exercise" zones

Figure 9. Optimal Exercise boundary for an American put with the primal DPG method.

derivative, finding the optimal exercise boundary for an American option is essential. The DPG method proposed in this section can find the optimal exercise boundary implicitly thanks to the projection-based method just by checking the price with the payoff at each moment or through an automatic procedure in the first active points at each time step in the primal-dual active set strategy. Fig. (9a) depicts finding this free boundary for the different interest rates of the market at each time to maturity. This optimal boundary is a powerful indicator for practitioners to choose the appropriate positions due to the hedging strategy. Thus, the optimal exercise boundary partitions the domain of the problem into an "Exercise region" and "Do not Exercise" region (9b) where the owner of the option will exercise the option when the stock price is at the green region, and will await in the red region.

5.3. *Asian Options*

Asian options can be classified as path-depended financial derivatives where the payoff of the option depends on the time average of the underlying security over some period of time such as the lifetime of an option Shreve (2004), Kemna and Vorst (1990). This average can be taken over continuous sampling or discrete sampling and the type of average can be an arithmetic average or geometric average. The closed-form value of an Asian option is not in hand, so a numerical scheme is an essential remedy to find the value of an Asian option.

Seeking a closed-form solution such as the Laplace transform of the price for this path-dependent derivatives has been a popular approach Vorst (1992), and Turnbull and Wakeman (1991). However, the numerical implementation of the aforementioned methods is troublesome for low volatility cases Fu *et al.* (1999). The Monte Carlo method can be used for the numerical solution, where it is well–known that this method is computationally expensive Kemna and Vorst (1990), and Broadie and Glasserman (1996). Another popular approach is solving two dimensions in space PDE to find the value of an Asian option Ingersoll (1987), Vecer (2001), and Kim *et al.* (2014). Besides, Rogers and Shi Rogers and Shi (1995) proposed a reduction approach where solving one-dimensional PDE obtains the value of the desired Asian option. However, both one and two-dimensional PDEs are susceptible to oscillatory solution and can blow up through time due to existing small diffusion terms.

In this section, we propose the DPG method for pricing the option based on the Black-Scholes pricing framework. Assume the dynamic of the underlying asset satisfies in a geometries Brownian motion defined in (19), then the payoff of an Asian call option at maturity with the fixed-strike is following

$$U(T) = \max\{\frac{1}{T}\int_0^T S(t)dt - K, 0\} = \max\{\frac{1}{T}\int_0^T S(t)dt, 0\}, \tag{53}$$

based on the risk-neutral pricing theory, the price $U(t)$ of this Asian option at time $t \in [0,T]$ yields

$$U(t) = \mathbb{E}[e^{-r(T-t)}U(T)|\mathcal{F}_t], \qquad \forall t \in [0,T], \tag{54}$$

where expectation in (54) is a conditional expectation with respect to the filter $\mathcal{F}_t$ of the probability space $(\Omega, P, \mathcal{F}_t)$. Since the payoff defined in (53) depends on the whole path of stock price $S(t)$, the price of this option is a function of t, $S(t)$, and the evolution of value underlying security over the path. Thus, we extend the pricing model presented in previous sections for the European and American options by defining a second process

$$Y(t) = \int_0^t S(v)dv, \tag{55}$$

where the dynamic of this new process $Y(t)$ follows a differential equation as following

$$dY(t) = S(t)dt. \tag{56}$$

Therefore, the value of the Asian option is also a function of $Y(t)$, so we denote the price of the Asian option with $U(t, S_t, Y(t))$. This function satisfies $\forall t \in [0, T]$, and $\forall (x, y) \in \mathbb{R}^+ \times \mathbb{R}$ in the following two-dimension in space, partial differential equation(see Shreve (2004), Kemna and Vorst (1990) for details)

$$\begin{cases} \dfrac{\partial U(t,S,y)}{\partial t} + \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 U(t,S,y)}{\partial S^2} + rS\dfrac{\partial U(t,S,y)}{\partial S} + S\dfrac{\partial U(t,S,y)}{\partial y} - rU(t,S,y) = 0, \\ \\ U(t,0,y) = e^{-r(T-t)}\big(\dfrac{y}{T} - K\big)^+, \qquad t \in [0,T), \quad y \in \mathbb{R}, \\ U(T,S,y) = \big(\dfrac{y}{T} - K\big)^+, \quad S \geq 0, \quad y \in \mathbb{R}, \\ \lim\limits_{y \to -\infty} U(t,S,y) = 0, \quad t \in [0,T), \quad S \geq 0. \end{cases} \tag{57}$$

Now, let's define a new state variable

$$x = \frac{1}{S(t)}(K - \frac{1}{T}\int_0^t S(s)ds). \tag{58}$$

Then, it has been shown Rogers and Shi (1995), Ingersoll (1987) that the price of the Asian option satisfies the following nonlinear backward partial differential equation

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\sigma^2}{2} x^2 \dfrac{\partial U^2}{\partial x^2} - (\dfrac{1}{T} + rx)\dfrac{\partial U}{\partial x} = 0, \\ \\ U(T,x) = \max\{0, -x\}, \end{cases} \tag{59}$$

where the partial differential equation (59) is one dimensional PDE in space. Eq. (59) is a nonlinear partial differential equation of convection-diffusion type with a convection term that is a function of volatility and spatial variable x. Thus, this differential equation belongs to the family of convection dominant problems where the coefficient of the convection term (second-order derivative) can be a very small number in this model. As we mentioned earlier in this section, this small coefficient could imply an oscillatory behavior such that it can lead to numerical instability for the numerical scheme Douglas and Russell (1982). On the other hand, the efficiency and robustness of the DPG method for the convection-diffusion problem have been successfully shown for the family of the convection-dominated problems (Ellis *et al.* (2016), Chan *et al.* (2014), Chan (2013), Demkowicz and Heuer (2013) and the references therein).

Having the solution of Eq. (59), the value of an Asian option with strike price K and initial stock value S_0 can be computed as $S_0 U(0, {}^{K}\!/_{S_0})$. After using a truncated computational domain $x \in [-2, 2]$ for the Eq. (59) and change of variable $\tau = T - t$ in time, the system of partial differential equation (59) will build into the following form,

$$\begin{cases} \dfrac{\partial U}{\partial \tau} - \dfrac{\sigma^2}{2} x^2 \dfrac{\partial U^2}{\partial x^2} + (\dfrac{1}{T} + rx) \dfrac{\partial U}{\partial x} = 0, \quad \forall x \in [-2, 2], \quad \forall \tau \in [0, T], \\ \\ U(0, x) = \max\{0, -x\}, \\ U(\tau, 2) = 0, \\ \dfrac{\partial U^2}{\partial x^2}(\tau, -2) = 0. \end{cases} \tag{60}$$

So, the option value will be $S_0 U(T, {}^{K}/_{S_0})$. Using our convention for the DPG method, we can write the weak form for the Eq. (60) as following

$$\frac{d}{d\tau}(u, \mathbf{v}) + b^{\tau}(u, \mathbf{v}) = 0, \tag{61}$$

$$\tag{62}$$

where the bilinear form for primal formulation defines as

$$b^{\tau}_{\text{primal}}(\mathbf{u}, v) = (\frac{\sigma^2}{2} x^2 \frac{\partial}{\partial x} u, \frac{\partial}{\partial x} v)_{\Omega} + \big((\frac{1}{T} + (r+2)x)\frac{\partial}{\partial x} u, v)_{\Omega} - \langle \hat{q}, v \rangle_{\partial\Omega}, \tag{63}$$

with a set of trial and flux variables $\mathbf{u} = (u, \hat{q}) \in H(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, test variable $\mathbf{v} = v \in L^2(\Omega)$. Moreover, considering trail variables $\mathbf{u} = (u, \vartheta) \in L_2(\Omega) \times L_2(\Omega)$, and flux variables $\hat{\mathbf{u}} = (\hat{u}, \hat{\vartheta}) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, the bilinear form (61) for the ultraweak formulation reads

$$\begin{aligned} b^{\tau}_{\text{ultraweak}}((\mathbf{u}, \hat{\mathbf{u}}), \mathbf{v}) &= b^{\tau}_{\text{ultraweak}}(((u, \vartheta), (\hat{u}, \hat{\vartheta})), (v, \omega)), \\ &= -(\vartheta, \frac{\sigma^2}{2} x^2 \frac{\partial}{\partial x} v)_{\Omega} + (\vartheta, (\frac{1}{T} + (r - \sigma^2) x v)_{\Omega} - (u, \frac{\partial \omega}{\partial x})_{\Omega} - (\vartheta, \omega)_{\Omega} \\ &\quad + \langle \hat{u}, \omega \rangle_{\partial\Omega} + \langle \hat{\vartheta}, v \rangle_{\partial\Omega}. \end{aligned} \tag{64}$$

Now, using backward Euler approximation for time derivative and appropriate discrete test and trial space for DPG explained in the section 5.1, the discrete DPG formulation for the Asian option pricing problem reads

$$(u^{n+1} - u^n, \mathbf{v}) + \Delta\tau b^{\tau}_n(u^n, \mathbf{v}) = 0, \qquad \forall \mathbf{v} \in V_h. \tag{65}$$

We propose the following graph norm for ultraweak formulation and energy norm for primal DPG formulation to solve the valuing Asian option problem formulated by Eq. (65)

$$\text{Primal}: \|v\|_V^2 = \frac{1}{\Delta t}\|v\|^2 + \frac{1}{(\Delta t)^2}\|\sigma^2 \frac{\partial}{\partial x} v\|^2,$$

$$\text{Ultraweak}: \|\mathbf{v}\|_V^2 = \|(v,\omega)\|_V^2 \tag{66}$$

$$= \frac{1}{(\Delta t)^2}\|\sigma^2 \frac{\partial}{\partial x} v - (\frac{1}{T} + (r-\sigma^2)v - \omega\|^2 + \frac{1}{\Delta t}\|\frac{\partial}{\partial x}\omega\|^2.$$

Therefore, one can obtain the corresponding discrete operators

$$B_{ij} = b^{\tau}(u_i, \mathbf{v}_j), \qquad G_{ij} = (\mathbf{v}_i, \mathbf{v}_j)_v, \qquad l_i = (u_i, v). \tag{67}$$

However, it is worth mentioning that the above rectangle matrix B is a function of the spatial variable, and the induced inner product $(\cdot,\cdot)_v$ is formed by the associated norms (66) defined in the procedure of the DPG formulation. Thus, discrete DPG formulation of the Eq. (60) $\forall n \in \{1,\cdots,N_\tau\}$, yields

$$\begin{cases} B^T G^{-1} l(u^{n+1} - u^n) + \Delta\tau B^T G^{-1} B u^n = 0, \\ \\ u^0 = \max\{0, -x\}, \qquad \forall x \in [-2,2] \\ u^n|_{x=2} = 0, \qquad \forall n \in 1,\cdots,N_\tau, \end{cases} \tag{68}$$

Thus, we can define near the optimal discrete DPG operators $B^{\text{n-op}} = B^T G^{-1} B$, $l^{\text{n-op}} = B^T G^{-1} l$ for the Eq. (60) for all $\forall n \in \{1,\cdots,N_\tau\}$, such that

$$\begin{cases} l^{\text{n-op}}(u^{n+1} - u^n) + \Delta\tau B^{\text{n-op}} u^n = 0, \\ \\ u^0 = \max\{0, -x\}, \quad \forall x \in [-2,2], \\ u^n|_{x=2} = 0, \qquad \forall n \in \{1,\cdots,N_\tau\}. \end{cases} \tag{69}$$

The system of Eq. (69) can be solved by an iterative method or a linear solver. In the next section, we examine the efficiency of the proposed DPG method.

5.4. *Numerical Experiments*

As mentioned before, the set of the partial differential equations (60) is a nonlinear and convection-dominant problem, and developing a numerical scheme for this problem can be problematic due to the convection term. In this section, we select some famous test problems from the literature to showcase the efficiency and accuracy of the proposed numerical scheme (69). In this example, all the results are generated by the first-order DPG method, and corresponding to the enriched test spaces ($\Delta p = 2$). We used $N_s = 100$ number of spatial elements, and the $N_t = 100$ time step for all the experiments in this section. Fig. (10) displays the value of the Asian option with two ultraweak and primal DPG formulations for different values of $\sigma = 0.05, 0.1, 0.2, 0.3$. As it can be seen the

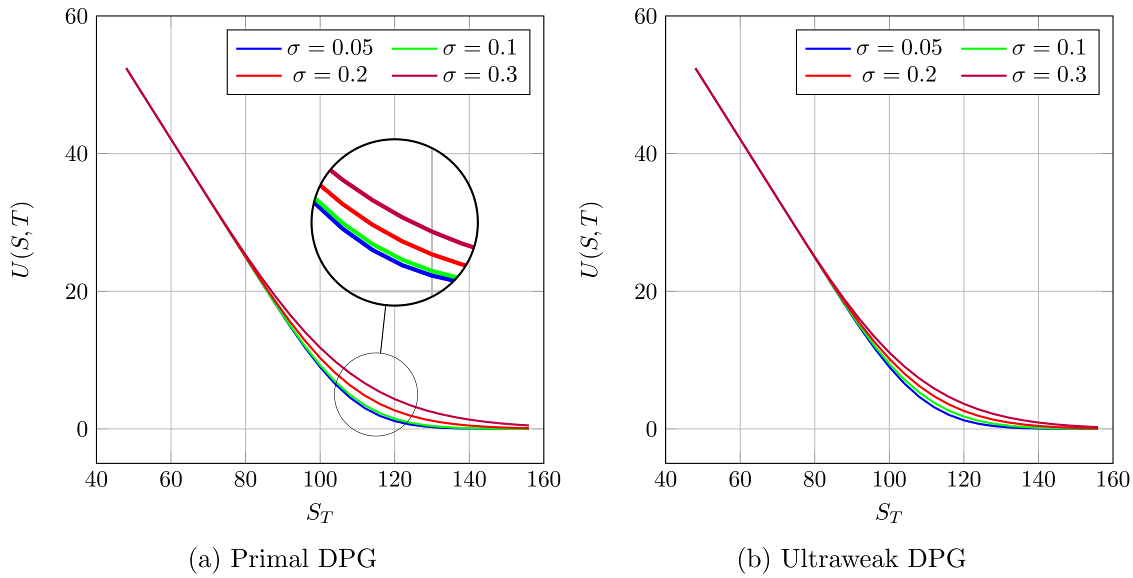


Figure 10. Value of an Asian option with DPG method for $r = 0.015$, and different volatilises

value of the Asian option is smooth and stable even for a small value of $\sigma = 0.05$ which leads to the convection-dominated case for the system of (69). We prepared the Table (2) to compare

Table 2. Asian call option with $r = 0.09$, $T = 1$, $S_0 = 100$

σ	Reference	Method	$K = 95$	$K = 100$	$K = 105$
	Zhang Zhang (2001)		8.8088392	4.3082350	0.9583841
	Zhang-AA2 Zhang (2003)		8.80884	4.30823	0.95838
	Zhang-AA3 Zhang (2003)		8.80884	4.30823	0.95838
0.05		Ultraweak DPG	8.8085332	4.3081967	0.958371
		Primal DPG	8.8088363	4.3082291	0.9583836
	ZhangZhang (2001)		8.9118509	4.9151167	2.0700634
	Zhang-AA2Zhang (2003)		8.91171	4.91514	2.07006
	Zhang-AA3Zhang (2003)		8.91184	4.915126	2.07013
0.10		Ultraweak DPG	8.910986	4.915116769	2.0700633
		Primal DPG	8.9118498	4.9151265	2.0700634
	Zhang Zhang (2001)		9.9956567	6.7773481	4.2965626
	Zhang-AA2Zhang (2003)		9.99597	6.77758	2.745
	Zhang-AA3Zhang (2003)		9.99569	6.77738	4.29649
0.20		Ultraweak DPG	9.99565668	6.7773481	4.2965626
		Primal DPG	9.9956567	6.7773429	4.2965619
	ZhangZhang (2001)		11.6558858	8.8287588	6.5177905
	Zhang-AA2 Zhang (2003)		11.65747	8.82942	6.51763
	Zhang-AA3 Zhang (2003)		11.65618	8.82900	6.51802
0.30		Ultraweak DPG	11.6558853	8.8287498	6.51779047
		Primal DPG	11.6558857	8.8287580	6.51779054

the result of DPG methodology for pricing an Asian option with interest rate $r = 0.09$, $T = 1$, $S_0 = 100$, different strike price $K = 95, 100, 105$, and different volatility with the result report in Zhang (2001), Zhang (2003). Considering the result from Zhang (2001) as a benchmark with the PDE method, one can see that the obtained results from DPG ultraweak and primal method are so close (less than 0.001% deviation). To compare the accuracy and stability of the proposed method with the broader method in the literature, Table (3) is produced. In this test, the results from the Monte Carlo method are exploited as an exact solution. We compute the value of an Asian option for different strike prices $K = 95, 100, 105$, the interest rate of $r = 0.15$, time to maturity $T = 1$, initial asset value $S_0 = 100$, with different volatility $\sigma = 0.05, 0.1, 0.2, 0.3$. The result from the DPG methods is a maximum 0.001% deviation from the benchmark.

Table 3. Asian call option with $r = 0.15$, $T = 1$, $S_0 = 100$

σ	Reference	Method	$K = 95$	$K = 100$	$K = 105$
0.05	vecer Vecer (2001)	Monte Carlo	11.094	6.795	2.745
		FDM	11.094	6.795	2.745
	Rogers and Shi Rogers and Shi (1995)	Lower Bound	11.094	6.795	2.745
	Foufas and Larson Foufas and Larson (2008)	FEM	11.112	6.810	2.754
	Kim et al. Kim *et al.* (2007)	MPCM	11.093	6.79	2.78
		Ultraweak DPG	11.09398	6.79512	2.74481
		Primal DPG	11.09401	6.7948	2.74499
0.10	vecerVecer (2001)	Monte Carlo	15.399	7.028	1.418
		FDM	15.399	7.029	1.415
	Rogers and Shi Rogers and Shi (1995)	Lower Bound	15.399	7.028	1.413
	Foufas and Larson Foufas and Larson (2008)	FEM	15.416	7.042	1.422
	Kim et al. Kim *et al.* (2007)	MPCM	15.398	7.028	1.448
		Ultraweak DPG	15.3984	7.0277	1.41769
		Primal DPG	15.39899	7.02812	1.418001
0.20	vecerVecer (2001)	Monte Carlo	15.642	8.409	3.556
		FDM	15.643	8.412	3.560
	Rogers and ShiRogers and Shi (1995)	Lower Bound	15.641	8.408	3.554
	Foufas and Larson Foufas and Larson (2008)	FEM	15.659	8.427	3.570
	Kim et al. Kim *et al.* (2007)	MPCM	15.66437	8.421	3.573
		Ultraweak DPG	15.64218	8.4091	3.5559
		Primal DPG	15.641865	8.4102	3.5584
0.30	vecerVecer (2001)	Monte Carlo	16.516	10.210	5.731
		FDM	16.516	10.215	5.736
	Rogers and ShiRogers and Shi (1995)	Lower Bound	16.512	10.208	5.728
	Foufas and LarsonFoufas and Larson (2008)	FEM	16.553	10.231	5.750
	Kim et al. Kim *et al.* (2007)	MPCM	16.5179	10.2194	5.742
		Ultraweak DPG	16.51615	10.21045	5.73074
		Primal DPG	16.51617	10.20964	5.730865

5.5. *Barrier Options*

A double knock-out Barrie option is a financial contract that gives a payoff $h(s)$ at maturity T, as far as the price of the underlying asset stays in the predetermined barriers $[S_L(t), S_U(t)]$, otherwise, if the spot price is hit barriers, the option gets knocked out. Although the barriers are checked continuously in time, it is more feasible to check the barriers discretely in the real–world application Shreve (2004).

It is well–known that the closed–form analytical solution for the discrete double barrier option is not known, so devising accurate and efficient numerical methods for valuing this type of option is essential. Thus, over the past years, researchers try to develop semi-analytical and numerical schemes for approximating the price of Barrier options. Here, we briefly address some of them. Kunitomo et.al Kunitomo and Ikeda (1992) used sequential analysis to find the solution as a series, analytical approach by contour integration is used by Pelsser Pelsser (2000) to price the barrier options. The binomial method is used by Cheuk et. al in Cheuk and Vorst (1996), and the Monte Carlo method as a probability-based method is devised in Ndogmo and Ntwiga (2007) to price this exotic option. PDE method, such as the finite difference method by Zevan et. al in Zvan *et al.* (2000), a finite element in Golbabai *et al.* (2014) by Golbabai et.al, and quadrature method in Milev and Tagliani (2010) have been developed for the pricing discrete barrier options.

We begin by stating the model of the problem which is inspired by the work Milev and Tagliani (2010), and Tse *et al.* (2001). Assume that dynamic of the underlying asset $\{S_t\}_{t\in[0,T]}$ is following the stochastic differential equation in (19), with the standard winner process W_t, interest rate r, volatility of σ, and fixed initial asset price S_0. Defining the Brownian motion Z_t, with instantaneous drift value $\hat{r} = r - (\sigma^2/2)$, and standard deviation σ, the price process will follow $S_t = S_0 e^{Z_t}$. Moreover, we define the discrete counterpart process $\tilde{X}_n = S_0 e^{\Theta_n}$, for $n = 1, 2, \cdots, N$, and $\Theta_n = \theta_1 + \theta_2 + \cdots + \theta_n$, $\Theta_0 = 0$. Random variables θ_i are independent normally distributed random variables i.e. $N(r - \sigma^2/2, \Delta t\sigma)$ with $\Delta t = \frac{T}{N}$ for N predetermined monitoring instants.

Consider the discrete monitoring dates of $t_1 = 0 \leq t_2 \leq \cdots \leq t_N = T$ with the constant upper and lower barriers of S_U, and S_L respectively. Besides, we assume that barriers are not active on the first, and last dates of our time interval. The price of a discrete double barrier option can be computed by discount of expected payoff at expiration time T to the present time t as follows.

$$e^{-rT}E[h(S_T)|\chi_{B_1}\chi_{B_2}\cdots\chi_{B_n}],$$

where the indicator functions of $\chi.$ is evaluating on sub set of $B_i = \{S_i \in (S_L, S_U)\}$.

Denoting $U(t,S)$ the value of a discrete double barrier option with the date of maturity of T, strike price K, this value will satisfy in the following system of N partial differential equations

$$\begin{cases} \dfrac{\partial U(t,S)}{\partial t} + \dfrac{\sigma^2}{2}S^2\dfrac{\partial^2 U(t,S)}{\partial S^2} + rS\dfrac{\partial U(t,S)}{\partial S} - rU(t,S) = 0, \quad \forall t \in [t_i, t_i+1], \quad \forall i = 1,2,\cdots,N \\ \\ U(0,t) = 0, \\ U(S,T) = \max\{S-K, 0\} \\ U(S,t_{i+1}) = h_{i+1}(S), \qquad \forall i = 1,2,\cdots,N \\ \lim\limits_{S\to+\infty} U(t,s) = h(S), \end{cases} \tag{70}$$

where boundary conditions $h_i(S)$, and $h_T(S)$ are also defined as

$$h_i(S) = \begin{cases} \lim\limits_{t\to t_i^+} U(S,t), & \text{if} \quad S_L \leq S \leq S_U, \quad \forall t \in [t_i, t_i+1], \quad \forall i = 1,2,\cdots,N \\ \\ 0, & \text{if} \quad S = \mathbb{R}^+ \backslash [S_L, S_U], \end{cases} \tag{71}$$

, and

$$h(S) = \begin{cases} S - Ke^{-r(T-t)}, & \text{if} \quad \forall t \in [t_{N-1}, t_N], \\ 0, & \text{if} \quad t \in [t_i, t_{i+1}], \quad \forall i = 1,2,\cdots,N-2 \end{cases} \tag{72}$$

As we can observe, the set of partial differential equations (70) is a system of consecutive partial differential equations where on each time interval $[t_i, t_{i+1}]$ has the final boundary conditions of (71), and the final PDE has the boundary condition (72). Besides, the system of PDEs presented in (70) with the aforementioned boundary condition is a nonlinear partial differential equation associated with the functions (71), (72), therefore, designing an accurate and stable numerical scheme is tricky here.

We use the change of variable in space and time similar to the change of variables for vanilla options in section 4.1 to obtain the following piecewise constant coefficient partial differential equations.

$$\begin{cases} \dfrac{\partial u(\tau,x)}{\partial \tau} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u(\tau,x)}{\partial x^2} + (r+\dfrac{\sigma^2}{2})\dfrac{\partial u(\tau,x)}{\partial x} - ru(\tau,x) = 0, \quad \forall \tau \in [\tau_i,\tau_{i+1}], \quad \forall i = 1,2,\cdots,N \\ \\ u(0,\tau) = 0, \\ u(x,0) = \max\{S-K,0\}, \\ u(x,\tau_i) = h_i(x), \qquad \forall i = 1,2,\cdots,N, \\ \lim_{x\to+\infty} u(\tau,x) = h(x). \end{cases} \tag{73}$$

Now if we concentrate on one of the equations as a generic differential equation on the interval $[\tau_j,\tau_{j+1}]$, where $j \in \{1,2,\cdots,N\}$, we propose the following weak formulation for DPG formulation

$$\frac{d}{d\tau}(\mathbf{u},\mathbf{v}) + b^{\tau}(\mathbf{u},\mathbf{v}) = 0, \qquad \forall \tau \in [\tau_j, \tau_j+1], \tag{74}$$

where the bilinear form is similar to the primal and ultraweak formulation defined in Eq. (44), and (45) on this sub–interval. However, the boundary conditions introduced (73) are performing on the interval $[\tau_j,\tau_{j+1}]$ as a sub–interval of the computational domain. Utilizing a generic partition $\tau_j = \tau_{j1}, \tau_{j2}, \cdots, \tau_{jN_j} = \tau_{j+1}$, for each interval, and using backward Euler scheme for time derivative, the approximate of Eq. (74) in the finite dimension space, the discrete DPG for each sub–partial differential equations reads

$$(u^{n+1} - u^n, \mathbf{v}) + \Delta\tau_i b_n^{\tau}(u^n,\mathbf{v}) = 0, \qquad \forall \mathbf{v} \in V_h., \quad \forall n \in \{1,2,\cdots,N_i\}, \tag{75}$$

where the time steps on the domain of each sub–problem defined as $\Delta\tau_\ell = \frac{\tau_{\ell+1}-\tau_\ell}{N_\ell}$, $\ell \in \{1,\cdots,N\}$. Indeed, on each problem (75) we need to solve a nonlinear discrete system of equations (see the Algorithm 1). Defining the graph and energy norm defined in (35) for each sub–domain $[\tau_j,\tau_{j+1}]$, and denoting the discrete operators of B, G, and l accordingly as following

$$B_{ij} = b^{\tau}(u_i,\mathbf{v}_j), \qquad G_{ij} = (\mathbf{v}_i,\mathbf{v}_j)_V, \qquad l_i = (u_i,\mathbf{v}_j)_V, \tag{76}$$

One can find the discrete nonlinear generic problems on each sub-domain $[\tau_i,\tau_{i+1}]$

$$\begin{cases} l^{\text{n-op}}(u^{n+1}-u^n) + \Delta\tau B^{\text{n-op}} u^n = 0, \qquad \forall n \in \{1,\cdots,N_{\tau_i}\}, \\ \\ u^n|_{x=0} = 0, \\ u^{N_{\tau_i}}|_x = h_i(x), \\ \lim_{x\to+\infty} u^n|_x = h(x), \end{cases} \tag{77}$$

where the near optimal DPG operators are defined as $B^{\text{n-op}} = B^T G^{-1} B$, $l^{\text{n-op}} = B^T G^{-1} l$. The system of equations of (77) can be solved by a projected iterative solver such as Gradient descent for different consecutive intervals till the time of maturity Beck (2014).

Algorithm 1 Numerical algorithm for the double barrier option

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Require: $S_0 \in [S_L, S_U]$
  $u^N|_S \leftarrow h(S)$
  for $\tau_i \in [t_1, t_N]$ **do**
    for $\tau_{i_j} \in [\tau_i, \tau_{i+1}]$ **do**
      if S is in $[S_L, S_U]$ **then**
        $u^n|_{S=0} = 0,$
        $u^{N_{\tau_i}}|_{S\to\infty} = h(S),$
        $u^{N_{\tau_i}}|_S = h_i(S),$

        Solve the sub-partial differential equation 77.

      else if S is out of $[S_L, S_U]$ **then**

        The option will be knocked out!

      end if
    end for
  end for

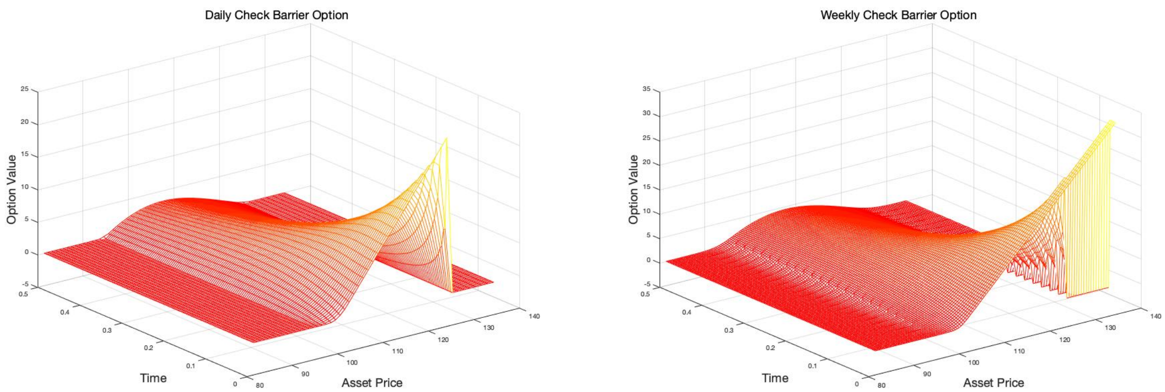
5.6. *Numerical Experiments*

Here we solve the standard test problem solved in Kim *et al.* (2014) problem. We use the DPG method to price a barrier option with volatility $\sigma = 0.2$, interest rate $r = 0.1$, strike price $K = 100$, and upper and lower boundary of $S_L = 95$, and $S_U = 125$ respectively. It is known that a trading year includes 250 working days, and a working week has five days. In this example, we report the numerical estimate for daily and weekly monitoring. In another words, if we take $T = 1$ (half year $T = 0.5$) for one trading year, then time increments of $\Delta t = 0.004$ (half year $\Delta t = 0.002$) corresponds with daily check and $\Delta = 0.02$ (half year $\Delta t = 0.01$) corresponds to weekly check. Using the first-order DPG method with $N_s = 100$ spatial element, $N_t = 100$ stepping time, and enriched test space with $\Delta p = 2$, the desired results will accomplish.

Fig. (11) depicts the surface of the price of the barrier option with the two primal and ultraweak formulations, as we expect this option is cheaper than the European option due to the convenience that brings for the trader. Moreover, in spite of the non–smooth boundary condition the surface of the price is smooth and stable.

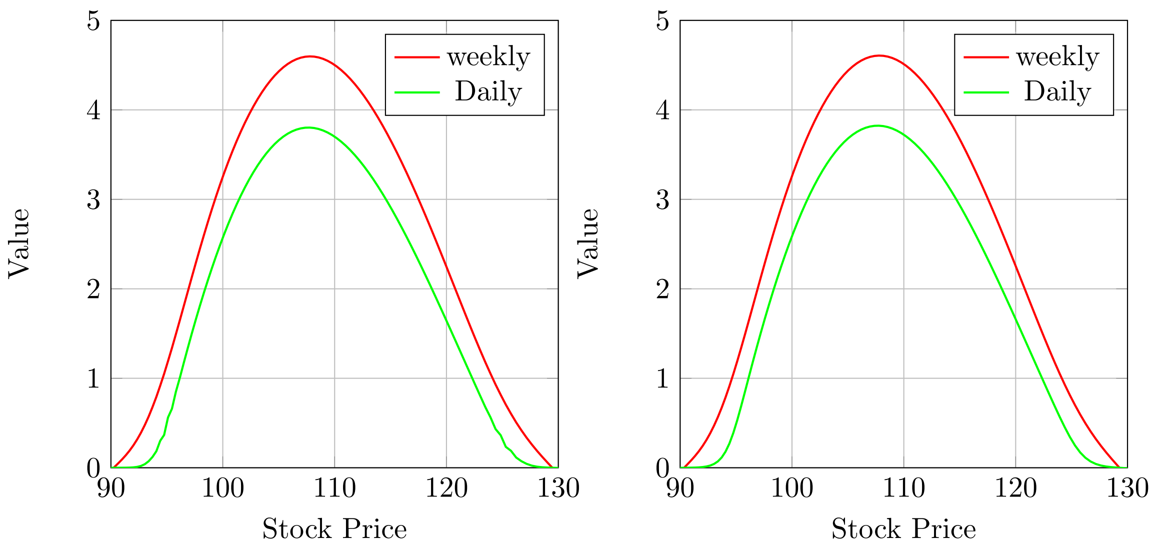
We prepared Fig. (12) to show the price of the barrier option with the aforementioned market parameters. The primal and ultraweak formulation is implemented to find the value of the option by checking both weekly and Daily for the barriers. One difficulty in pricing barrier options is that the value of the option can be oscillatory near the barriers of S_L, and S_U, whereas the illustrations show the stable and smooth behavior of the price for the value of stock price close to the boundaries.

Table (4) compares the accuracy of the DPG method with the path integral method Milev and Tagliani (2010), and MPCM method Kim *et al.* (2007). In this experiment, we see the value of the option for daily and weekly monitoring when the price of the underlying asset is $S = 95$, $S = 95.0001$, $S = 124.9999$, and $S = 125$ when barriers are $[S_L, S_U] = [95, 125]$. As we mentioned before the numerical scheme can have unstable behavior close to barriers, and in this example, we try to catch the accuracy of the method when the stock price is in a very close neighborhood of barriers. As can be seen, the DPG method is accurate and very close to the recorded value in Milev and Tagliani (2010), and Kim *et al.* (2007).



Primal DPG Ultraweak DPG

Figure 11. Surface of the price of barrier option, $\sigma = 0.2$, r =0.1, K=100, $[S_L, S_p] = [95, 125]$ via DPG method.



(a) Primal DPG (b) Ulraweak DPG

Figure 12. Value of Barrier option with DPG method

6. Sensitivity Analysis with Greeks

In this section, we use the DPG methodology to calculate the sensitivity of option pricing under the Black-Scholes model. Sensitivity of the option with respect to model parameters, Greeks, explains the reaction of the option value to the fluctuation of the market environment. Greeks are compasses in the trader's hand to find the correct direction in the hope of hedging their portfolio by buffering against market changes.

Thus, the efficiency and accuracy of the numerical scheme are of paramount importance to trace the option price changes when the state of the market changes. Let $u(s,t)$ be the solution of Black-Scholes partial differential (21) with the appropriate boundary condition pertaining to that specific option, and α is the desired parameter for which we want to see the changes of price, then $\frac{\partial u(x,t)}{\partial \alpha}$ which for simplicity it will be denoted by $u_\alpha(x,t)$ is the sensitivity. This sensitivity can be found with the direct method or dual method (the avid readers can see Damircheli and Bhatia (2019)).

Table 4. Double Barrier option with $\sigma = 0.2$, $r = 0.1$, $T = 0.5$, $K = 100$, $L = 95$, $U = 125$

S	Reference	Method	Weekly checking	Daily checking
95	Milev Milev and Tagliani (2010)	Path integral	1.04584	0.444389
	Kim Kim *et al.* (2007)	MPCM	1.04581	0.444374
		Primal DPG	1.04584	0.444387
		Ultraweak DPG	1.04583	0.444388
95.0001	Milev Milev and Tagliani (2010)	Path integral	1.04584	0.444426
	Monte Carlo	10^7 paths	1.0457	0.44431
	Kim Kim *et al.* (2007)	MPCM	1.04584	0.444412
		Primal DPG	1.04583	0.444413
		Ultraweak DPG	1.04583	0.444412
124.9999	Milev Milev and Tagliani (2010)	Path integral	0.70735	0.284354
	Monte Carlo	10^7 paths	0.70731	0.28442
	Kim Kim *et al.* (2007)	MPCM	0.707343	0.284354
		Primal DPG	0.707334	0.284352
		Ultraweak DPG	0.707336	0.284354
125	MilevMilev and Tagliani (2010)	Path integral	0.707313	0.284337
	Kim Kim *et al.* (2007)	MPCM	0.707322	0.284334
		Primal DPG	0.707319	0.284333
		Ultraweak DPG	0.707321	0.284335

Taking the derivative with respect to the parameter α from Black-Scholes PDE (21, one can find a system of partial differential equation that seeks for $u_\alpha(x,t)$

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial \alpha}(\frac{\sigma^2}{2}x^2)\frac{\partial u^2}{\partial x^2} + \frac{\sigma^2}{2}x^2\frac{\partial u_\alpha^2}{\partial x^2} + \frac{\partial}{\partial \alpha}(rx)\frac{\partial u}{\partial x} + rx\frac{\partial u_\alpha}{\partial x} - \frac{\partial r}{\partial \alpha}u(x,t) - ru_\alpha = 0. \tag{78}$$

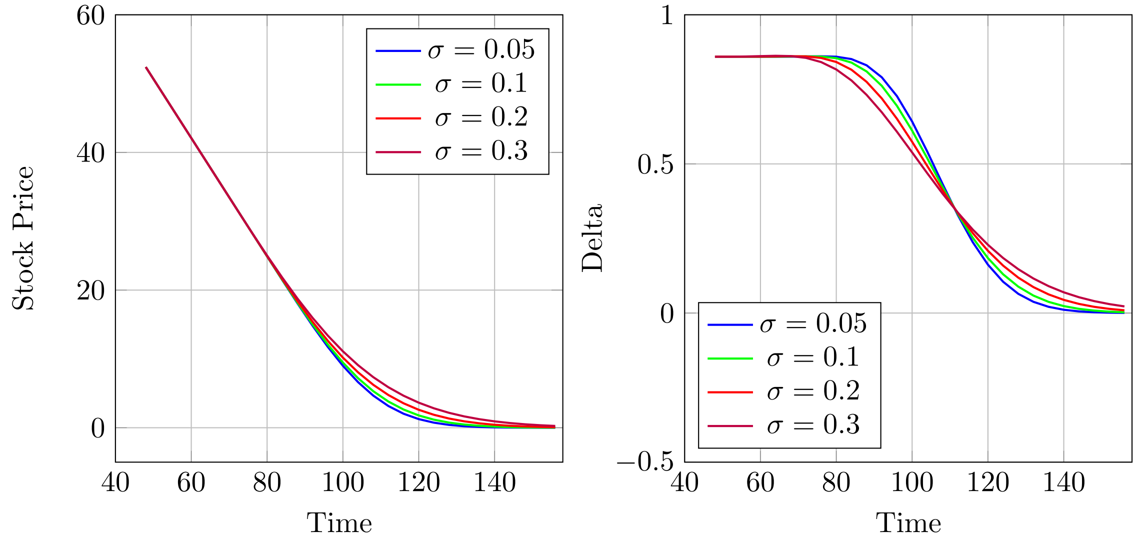
Note, the $u(x,t)$ is already evaluated the value of the option in the initial state of parameter α (see Seydel and Seydel (2006) for more detail). One can develop a DPG formulation either primal or ultraweak for solving the PDE presented in (78) to find the desired sensitivity of $u_\alpha(x,t)$ with appropriate boundary condition. In this paper, we study the first and second derivative of price with respect to the underlying asset that are named as Delta and Gamma respectively.

To start, it is worth mentioning that in the ultraweak formulation of DPG method (for example see (32)) inherently and implicitly we are evaluating the Delta since our primary trail variables are $(u(x,t), {}^{\partial u}/_{\partial x})$. Fig. (13) the numerical result of ultraweak solution of the Asian option pricing problem as an example is prepared to show how Delta can implicitly be calculated without extra computational cost for recalculation of sensitivity.

However, one can indirectly find the Gamma and Delta of Asian option with ultraweak formulation and primal formulation Fig. (14) using the PDE (78) for different volatility of the market.

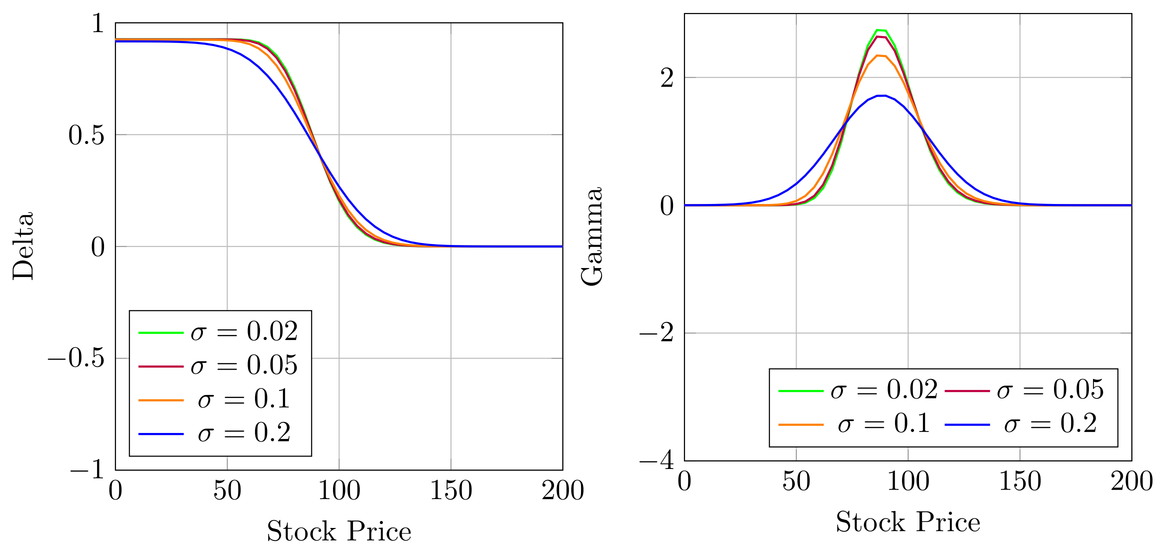
It is well–known that Delta is positive for call options Fig. (15a) and negative for put option Fig. (16a), whereas Gamma is always positive for both call options Fig. (15b) and put options Fig. (16b). Fig. (15) is prepared to illustrate Delta and Gamma of the European call option for different times to maturity, strike price $K = 100$, $r = 0.05$, and $\sigma = 0.15$ with primal DPG method. The sensitivity of the European put option with the same market parameters is depicted in Fig. (16) using the ultraweak DPG method.

Admittedly, the American option is one of the most attractive options for market makers since they have the right to exercise the contract once they find the appropriate moment based on their hedging strategy. Thus, not only the monitoring Delta is important, but practitioners are curious about the rate of change in Delta itself (Gamma) for each one-basis point movement in the underlying asset. However, we can expect that the free boundary attained by the early exercise feature has a significant impact on the sensitivity of the option as well. Fig. (17) shows the violation in Delta and Gamma for an American Put option based on the Primal DPG method in the different



(a) Value of Asian option (b) Delta for Asian option

Figure 13. Computing the Delta for Asian option alongside the value of the option with the ultra-weak DPG method for different volatilises

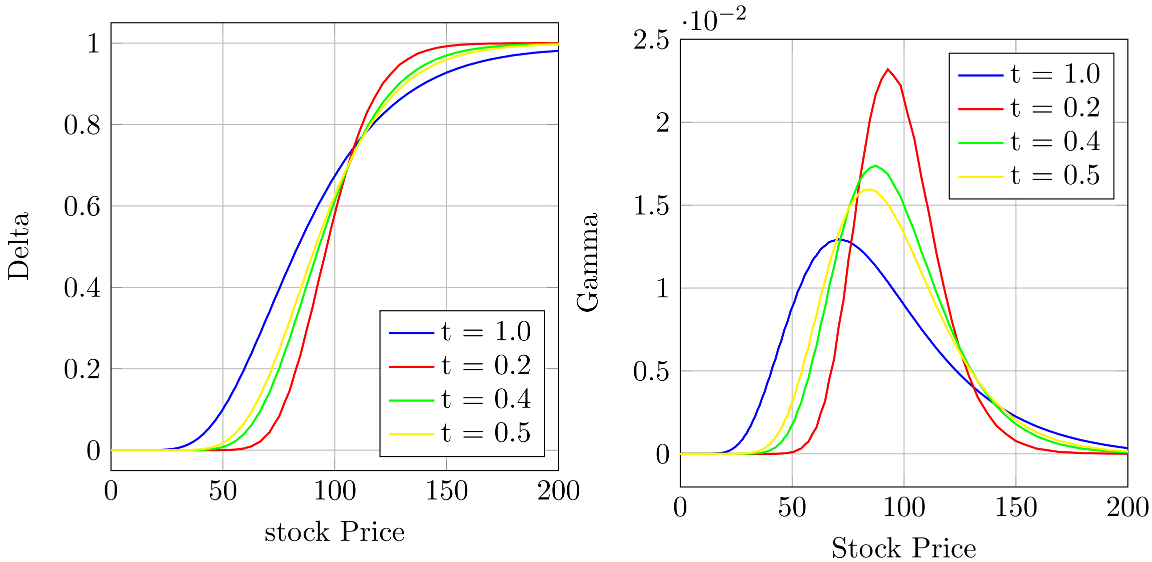


(a) Delta with Primal DPG (b) Gamma with Primal DPG

Figure 14. Greeks of Asian option with primal DPG method

time to maturities. As we can see this chaotic behavior as the time approaches maturity increases such that at $t = 0.01$ shortly after locking the option we have smooth behavior like the European option and at time $t = 1.0$ we have maximum fluctuation.

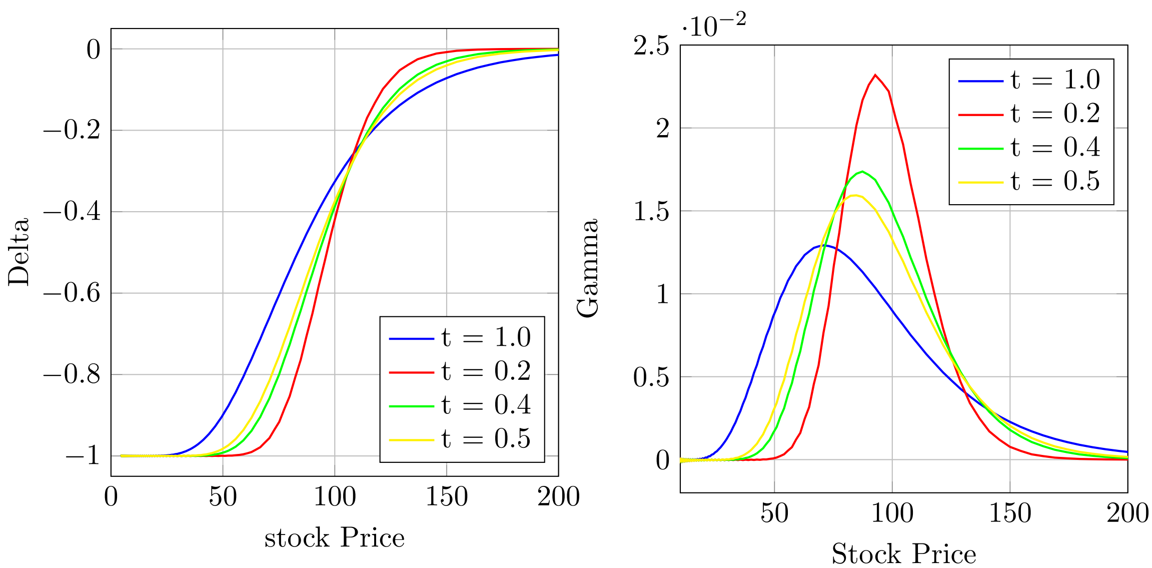
Greeks for barrier option with the double barrier $S_L = 95$, and $S_U = 125$ has shown in Fig. (18), and Fig. (19) using DPG method for different initial stock price $S_0 = 95.0001$, $S_0 = 100$. Both figures show that the sensitivity has sinusoidal behavior around the barriers when the underlying price is close to 95, and 125. One can see that in both cases rate of change in price and Delta are more smooth for weekly checking the barriers in comparison to daily check of the barrier which stands to reason.



(a) Delta with Primal DPG

(b) Gamma with Primal DPG

Figure 15. Greek of European call option with Primal DPG, r=0.05, σ=0.15, K=100



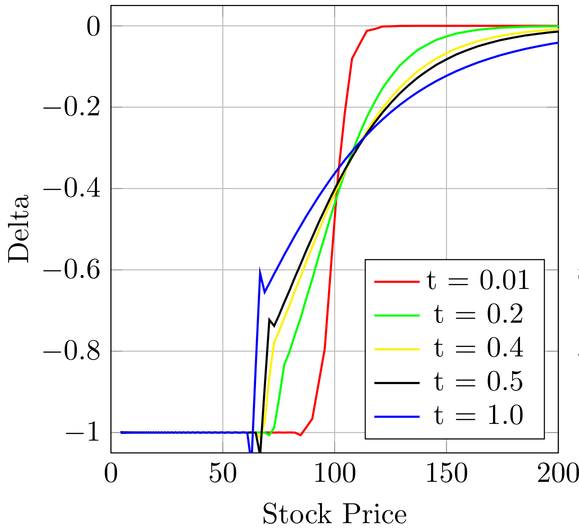
(a) Delta with Ultraweak DPG

(b) Gamma with Ultraweak DPG

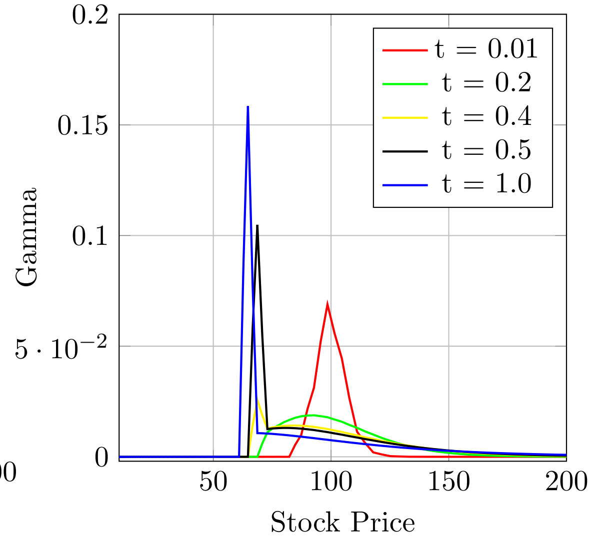
Figure 16. Greek of European Put option with ultraweak DPG, r=0.05, σ=0.15, K=100

7. Conclusion

In this manuscript, a numerical scheme based on the discontinuous Petrov–Galerkin (DPG) is proposed to deal with the option pricing problem as one of the most important branches of quantitative finance. The Black-Scholes PDE arisen from option pricing is a special member of the family of the convection-diffusion problem which is known for being unstable in the case of having a convention-dominant term. The DPG method automatically yields a stable numerical method to estimate the solution of the very same PDE. In this investigation, we derived detailed DPG formulations for European, American, Asian, and Barrier options, and their sensitivity. Besides, computational experiments is performed to inspect the numerical efficiency of the method for each option and corresponding Greek.

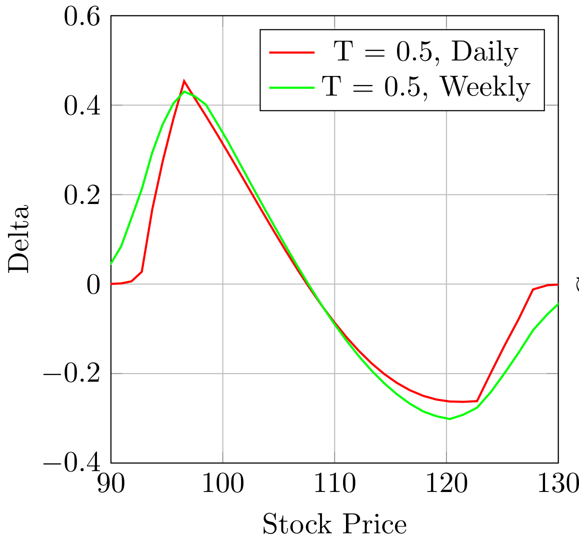


(a) Delta with Primal DPG

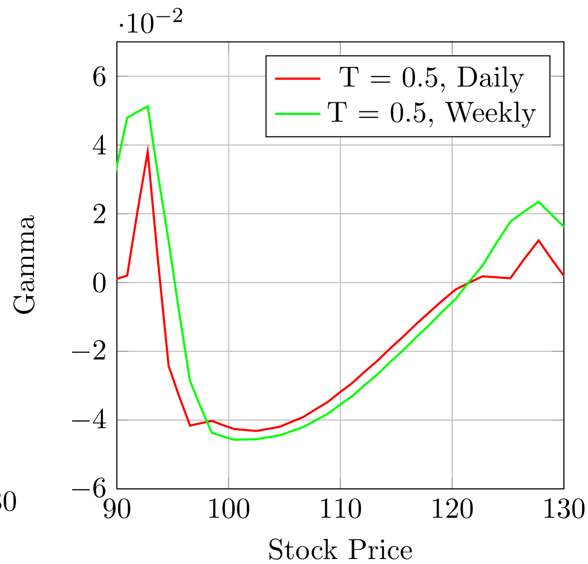


(b) Gamma with Primal DPG

Figure 17. Greeks of American Put option with primal DPG, r=0.05, σ=0.15,K=100



(a) Delta with Primal DPG



(b) Gamma with Primal DPG

Figure 18. Greeks of Barrier option for weekly and daily, p=2, Nt = 25, S0=95.0001

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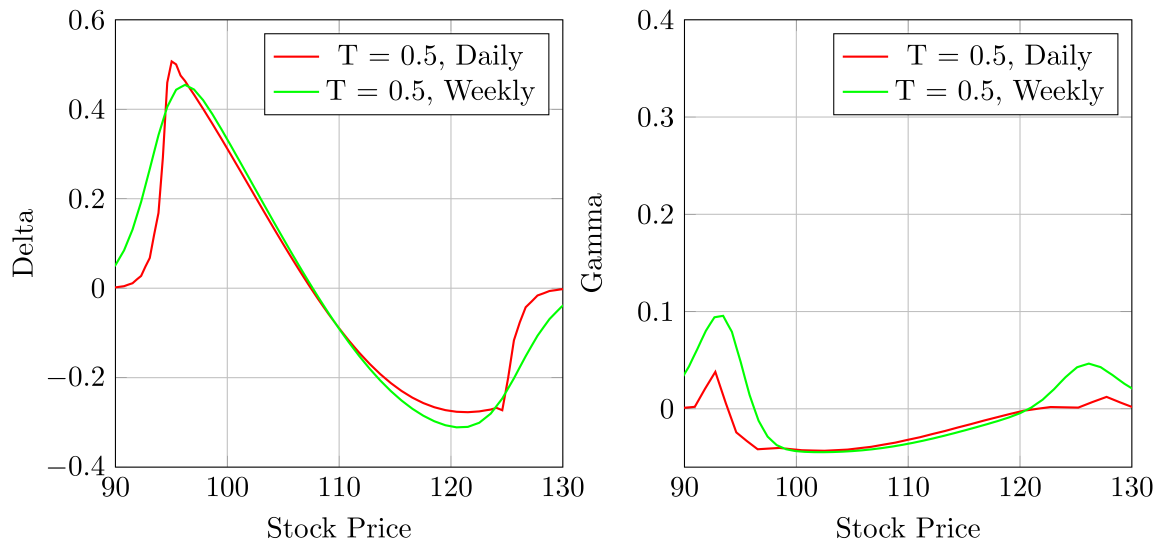

(a) Delta with Ultraweak DPG (b) Gamma with Ultraweak DPG

Figure 19. Ultraweak Barrier for weekly and daily, p=2, Nt = 25, S0 = 100

Disclosure of interest

There are no relevant financial or non-financial competing interests to report.

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